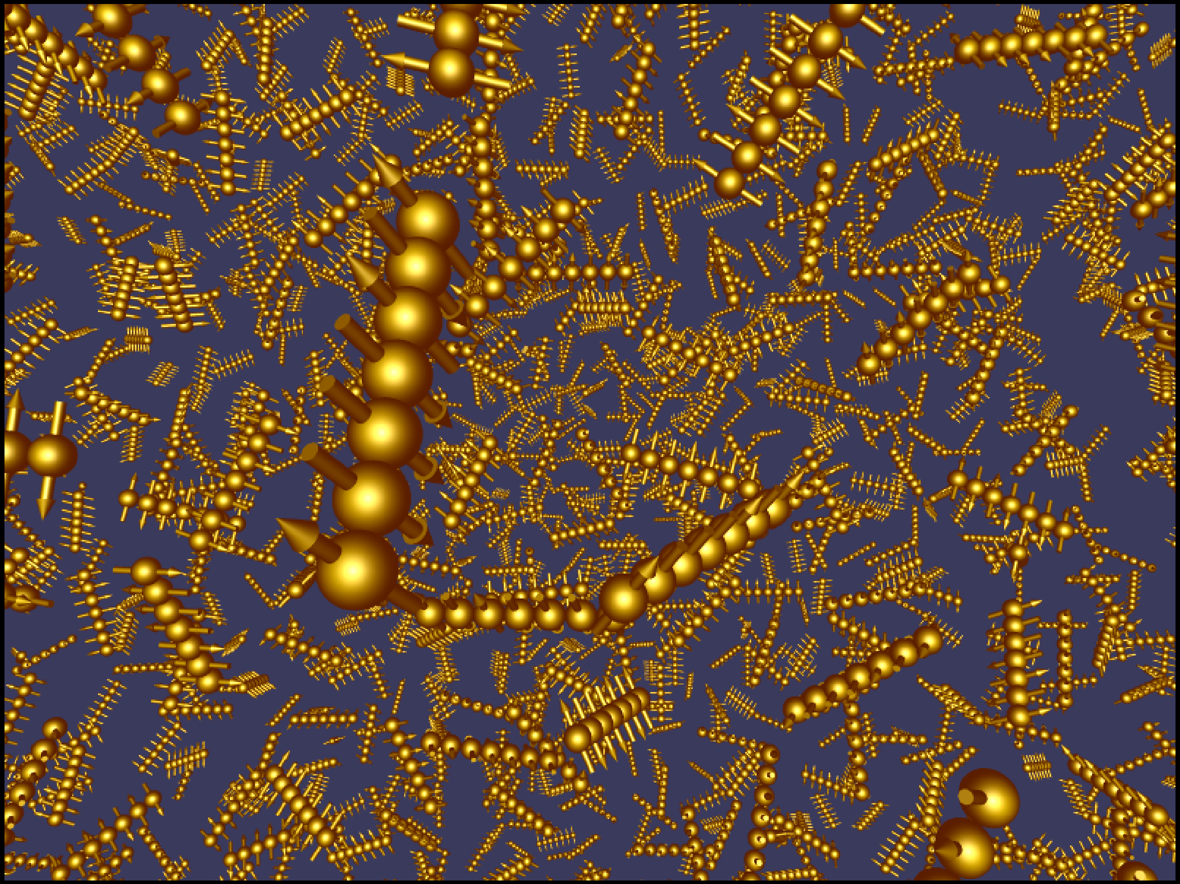
Quantum integrability and the entanglement spectrum

Zlatko Papic



“Localization in Quantum Systems”, King’s College London, 2/6/2017

UNIVERSITY OF LEEDS

This talk

(1) Where did it come from?

→ **Some universal properties of states of systems that avoid thermalization revealed by the entanglement spectrum**

[M. Serbyn, A. Michailidis, D. Abanin and ZP, Phys. Rev. Lett. **117**, 160601 (2016)]

(2) Does it represent some intelligent design?

→ **How "far" is the state from any free state?**

[C. Turner, K. Meichanetzidis, ZP, and J. Pachos, arXiv:1607.02679; Nat. Commun. 10.1038/ncomms14926 (2017)]

Entanglement spectrum

$$|\psi\rangle \to \quad \rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| \qquad S_A = -Tr_A \rho_A \log \rho_A$$

How does it scale with size of A?

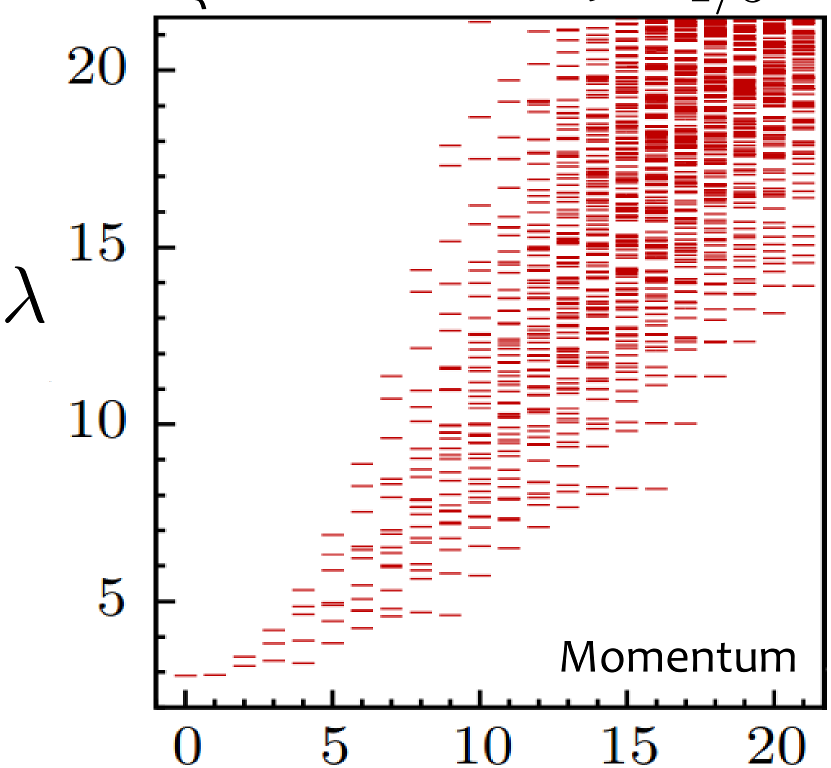
$S_A = 0$ product (unentangled) state

$S_A \propto \mathrm{vol}_A$ random (thermal) state

$S_A \propto \mathrm{area}_A - \gamma + \ldots$ "area law" (gapped states)

[Kitaev, Preskill; Levin, Wen '05]

Quantum Hall state $\nu = 1/3$



Entanglement spectrum
[Li, Haldane '08]

$$|\psi\rangle = \sum_k \sqrt{\lambda_k}|A_k\rangle|B_k\rangle$$

Is there more generic content?

(e.g., independent of geometric or conformal symmetries)

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Generic behavior of closed quantum systems

What is the generic behavior of isolated quantum many-body systems at arbitrary energy density?
(open problem even in 1D)

A useful probe: global quench

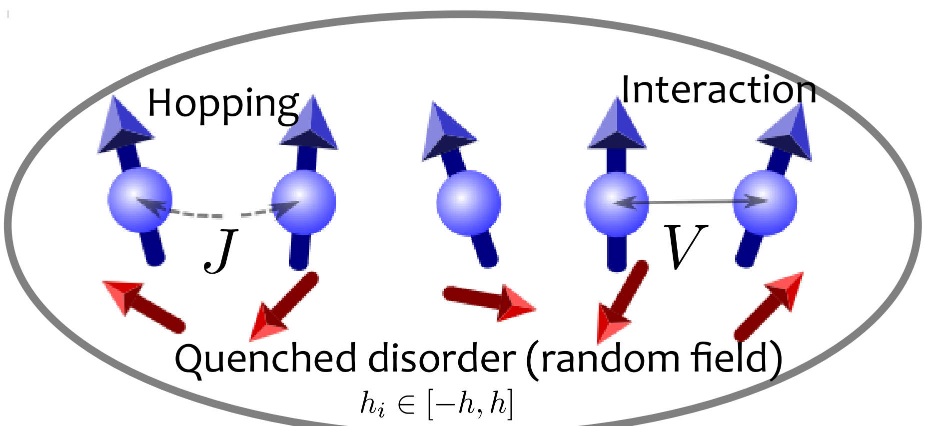
1. Prepare an unentangled initial state

$$\psi_0 = |\ldots \uparrow\downarrow\uparrow\downarrow \ldots\rangle$$

2. Evolve with a known Hamiltonian and observe

$$\psi(t) = e^{-itH}\psi_0$$

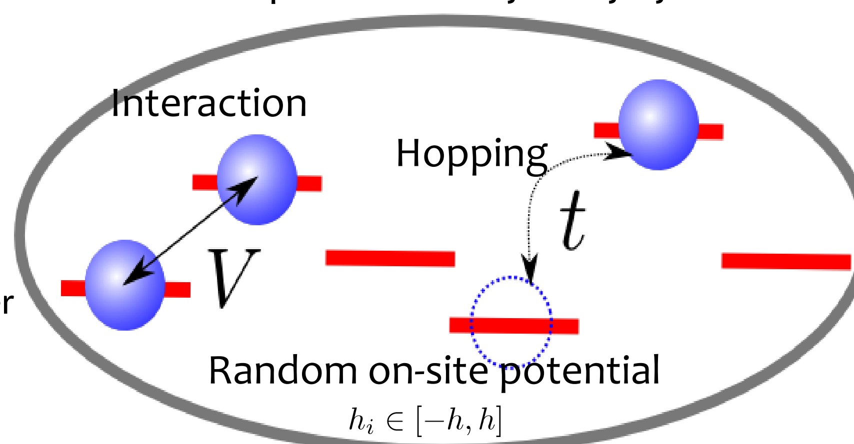
Isolated quantum many-body system



$\approx$

Jordan-Wigner

Isolated quantum many-body system



$$H = J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + V\sum_i \sigma_i^z\sigma_{i+1}^z + \sum_i h_i\sigma_i^z$$

$$H = t\sum_i c_i^\dagger c_{i+1} + h.c. + V\sum_i n_i n_{i+1} + \sum_i h_i n_i$$

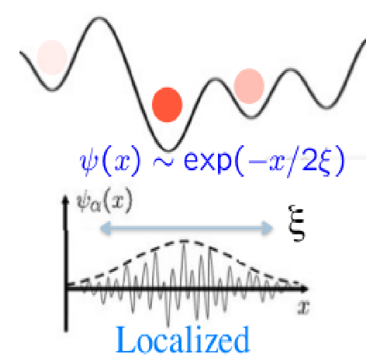
Dynamics of entanglement: Thermalization vs. Localization

$$H = J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + V\sum_i \sigma_i^z\sigma_{i+1}^z + \sum_i h_i\sigma_i^z$$

hopping | interaction | random field $h_i \in [-W, W]$

0 **ergodic** **"Many-body localized" phase** W

$W = \infty$
Anderson insulator

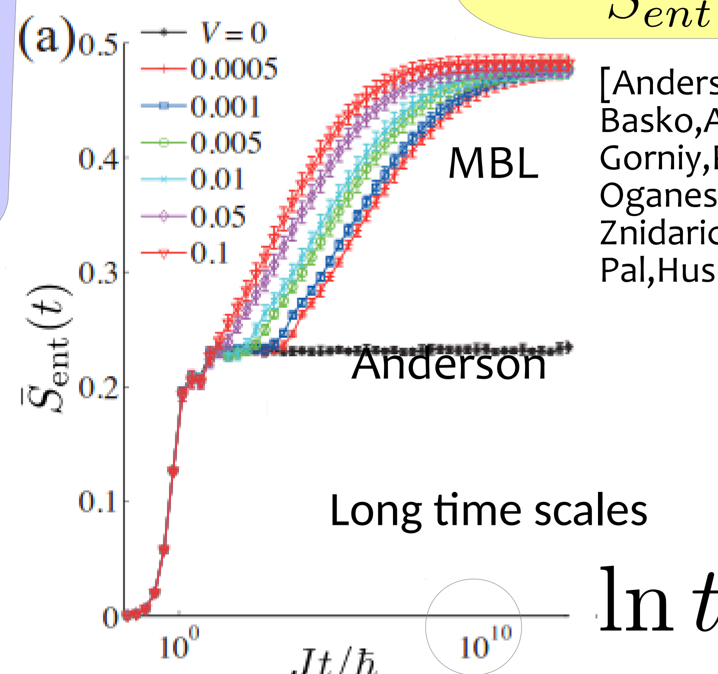


- An infinite system is a heat bath
- System achieves thermal equilibrium
- Finite D.C. transport
- Extensive entanglement

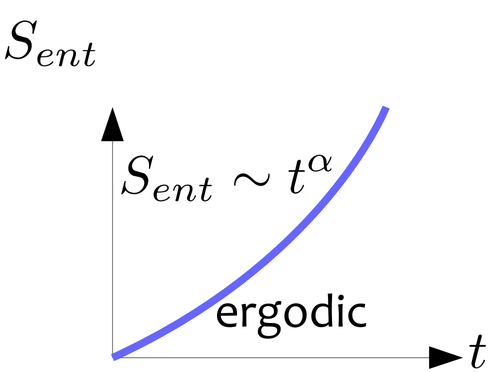
$S_{ent} \propto \mathrm{vol}_A$

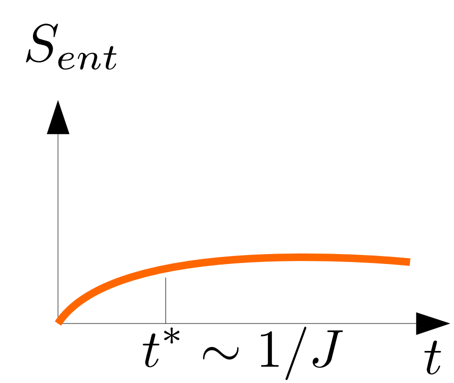
- Not a heat bath
- Doesn't thermally equilibrate
- D.C. transport is zero
- "Area-law" entanglement

$S_{ent} \propto \mathrm{area}_A$



[Anderson, Fleishman'80; Basko,Aleiner,Altshuler'05; Gorniy,Polyakov,Mirlin'05; Oganesyan,Huse'08; Znidaric,Prosen, Prelovsek '08; Pal,Huse'10, ...]





[Bardarson, Pollmann, Moore, '12; Serbyn, ZP, Abanin,'13]

Local integrals of motion in the MBL phase

Phenomenological Hamiltonian in the MBL phase:

$$H = \sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + \sum_i h_i \sigma_i^z$$

$$H = \sum_i h_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \ldots$$

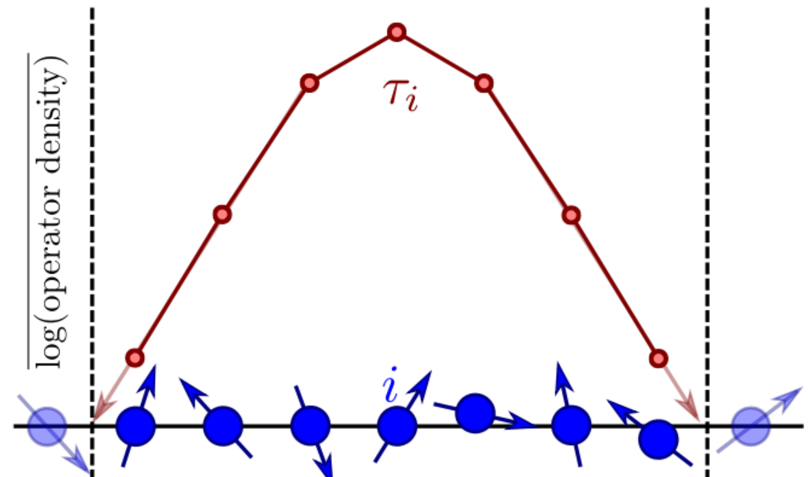
[Serbyn, ZP, Abanin, '13;
Huse, Nandkishore, Oganesyan, '13;
Imbrie '14; Chandran *et al.* '14;
Ros *et al.*, '14]

$$J_{ij} \propto e^{-|i-j|/\xi}$$

Local integrals of motion:

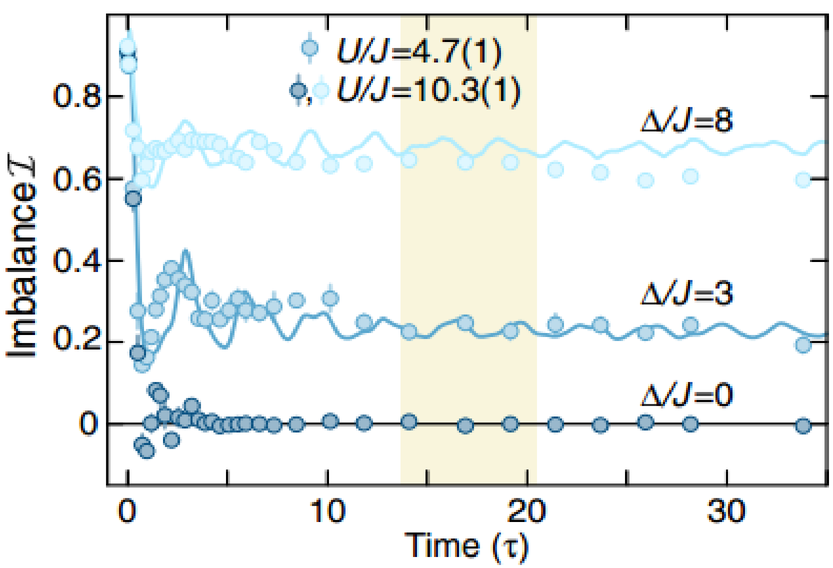
$$[\tau_i^z, H] = 0, \; [\tau_i^z, \tau_j^z] = 0$$

$$\tau_i^z \approx \sigma_i^z + f_{ab}^{i;jk} \sigma_j^a \sigma_k^b + \ldots$$



[T. O'Brien, D. Abanin, G. Vidal, ZP, arXiv:1608.03296;
see also Rademaker, Ortuno '15; You *et al*, '15; Inglis, Pollet '16;
ergodc phase: H. Kim, M. C. Banuls, J. I. Cirac, M. B. Hastings, and D. A. Huse, '15]

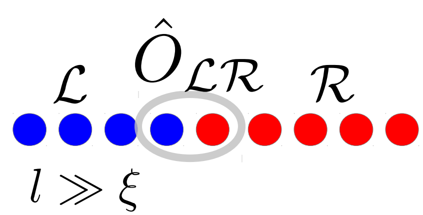
- **Consequence 1: System does not relax**



[M. Schreiber *et al*, Science **349**, 842 (2015)]

- **Consequence 2: Area law entanglement**

[Bauer, Nayak '13]



"MBL eigenstates are similar to ground states of gapped systems"

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[Serby
Huse,
Imbrie
Ros et

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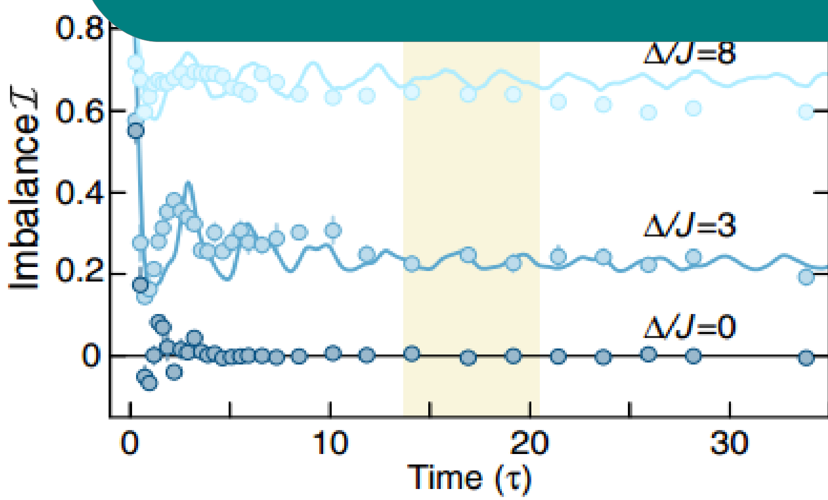
How do the local integrals of motion affect the entanglement?

Entropy obeys area law, what about entanglement spectrum?

•



[M. Schreiber *et al*, Science **349**, 842 (2015)]

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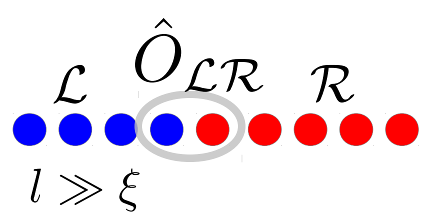

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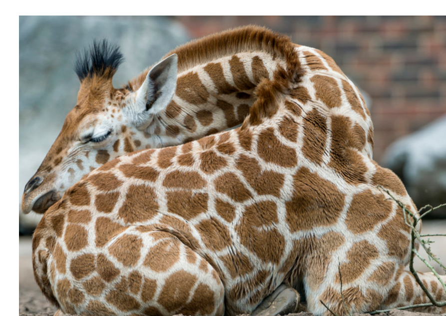


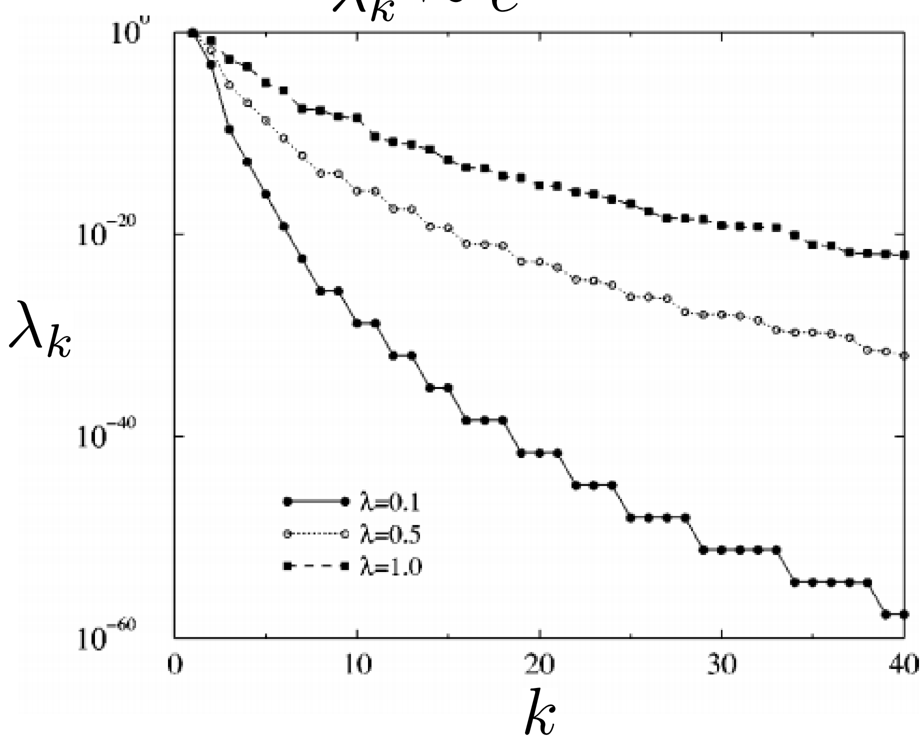
Known universal properties of the ES in generic systems

Free systems: $\rho = K e^{-\sum_l \epsilon_l f_l^\dagger f_l}$
Wick's theorem

[Peschel, Chung '01;
Okunishi, Hieida, Akutsu '99]

Example: quantum Ising model

$$\lambda_k \sim e^{-\alpha(\ln k)^2}$$



Critical systems in 1D

Entanglement spectrum has universal form given only by central charge of the CFT
[Calabrese and Lefevre '08; Pollmann and Moore, '09]

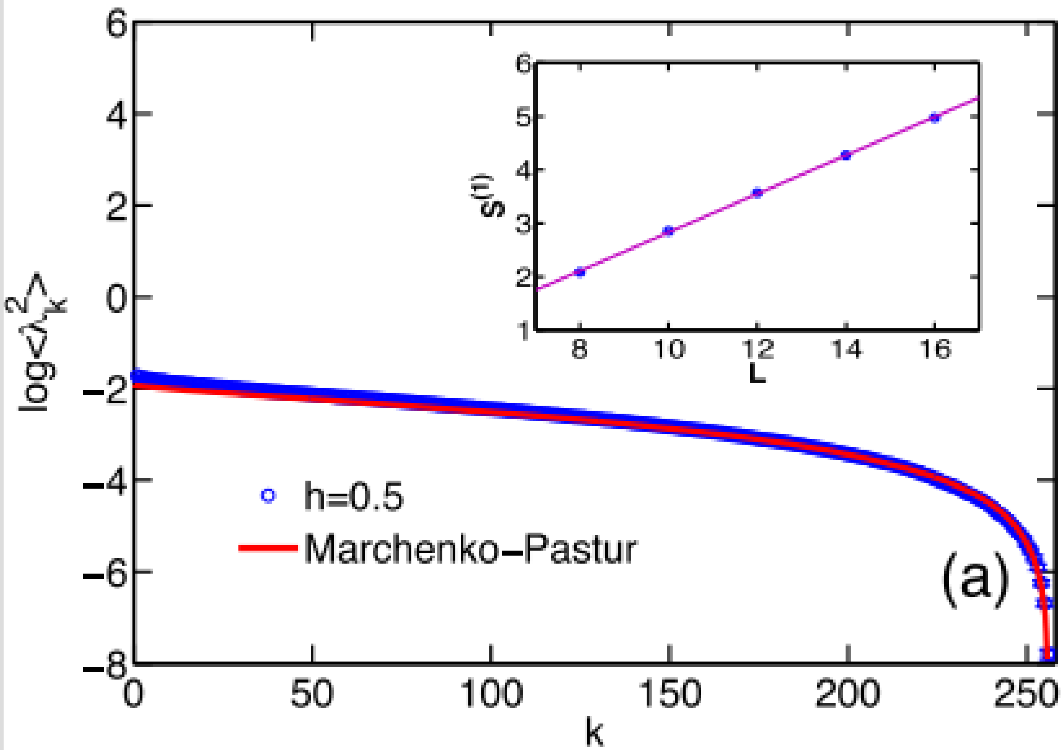
In a random or thermal state:

Marchenko-Pastur distribution

= density of eigenvalues of a Wishart matrix

$$Y = XX^\dagger$$

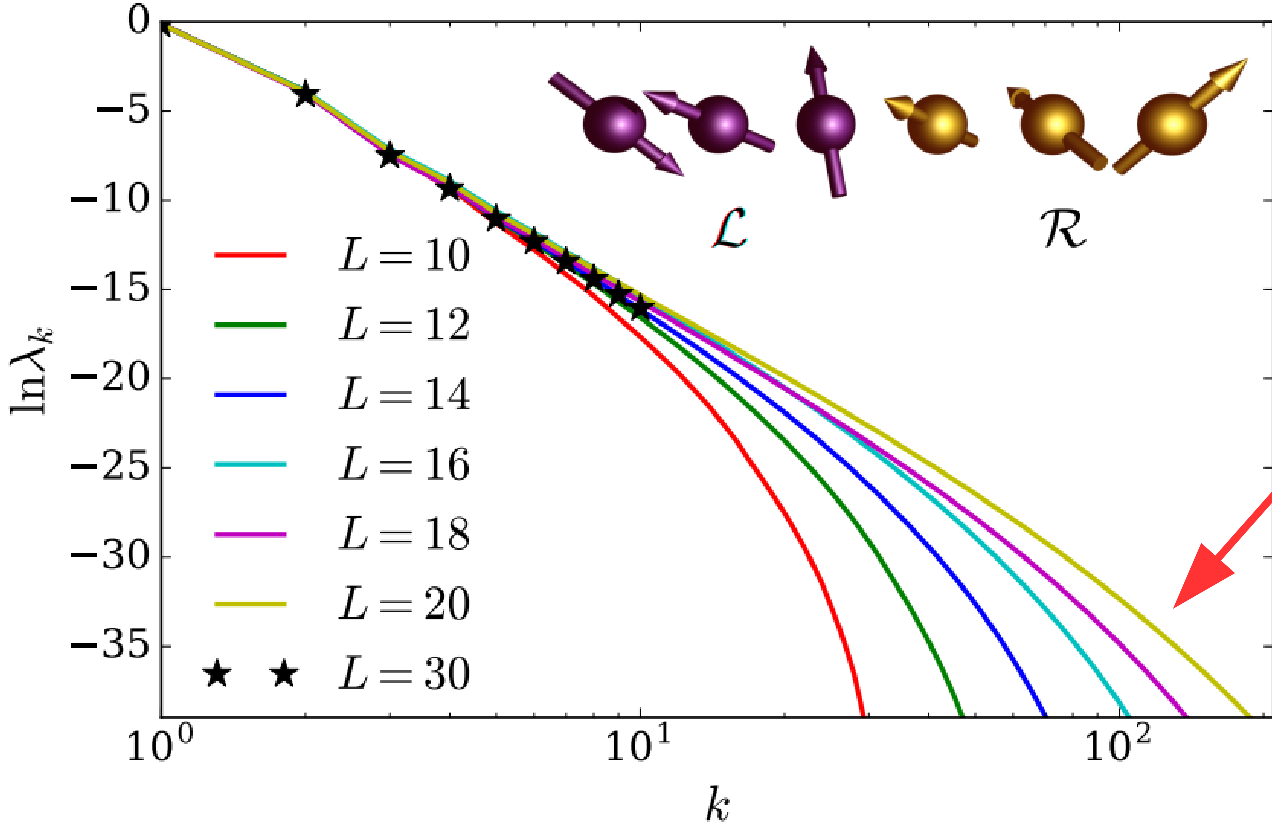
Example: Heisenberg model with small disorder



[Zhi-Cheng Yang *et al.*, PRL **115**, 267206 (2015)]

Power-law entanglement spectrum in MBL systems

$$\langle \ln \lambda_k \rangle \propto -\gamma \ln k \qquad \gamma = \frac{4\kappa}{\ln 2} \qquad \kappa = \xi^{-1}$$



[Related work on the statistics of spacing between Schmidt values: Geraedts, Nandkishore, Regnault, arXiv:1603.00880]

Finite size tails are described by "order statistics" for the Gaussian distribution

$$\ln \lambda_k \approx c_1 - c_2 \mathrm{erf}^{-1}\left(\frac{2k - 2^{L_A} - 1}{2^{L_A} - 2a + 1}\right)$$

[Blom, '58]

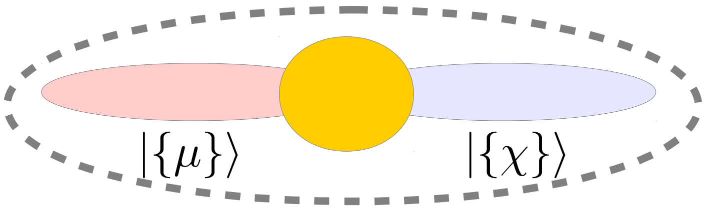
- There is no symmetry, so the **effective quantum number is Schmidt rank**
- **Different from typical ground states** of gapped systems where the ES decays faster
- Useful for **benchmarking MPS**-type variational calculations in MBL context

[M. Serbyn, A. Michailidis, D. Abanin and ZP, PRL **117**, 160601 (2016)]

Power law from local integrals of motion

Pick an eigenstate and expand it:

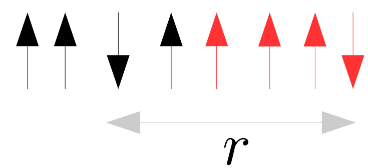
$$|I\rangle = \sum_{\{\mu\}_\mathcal{L},\{\chi\}_\mathcal{R}} C_{\{\mu\}_\mathcal{L}\{\chi\}_\mathcal{R}} |\{\mu\}_\mathcal{L}\rangle \otimes |\{\chi\}_\mathcal{R}\rangle$$



e.g. $\tau_i|I\rangle = |I\rangle$

$$\hat{\rho}_\mathcal{R} = \sum_{\{\mu\}_\mathcal{L}} |\psi_{\{\mu\}_\mathcal{L}}\rangle\langle\psi_{\{\mu\}_\mathcal{L}}|$$

$$|\psi_{\{\mu\}_\mathcal{L}}\rangle = \left(C_{\{\mu\}_\mathcal{L}\{\chi_1\}_\mathcal{R}}, \ldots, C_{\{\mu\}_\mathcal{L}\{\chi_{D_\mathcal{R}}\}_\mathcal{R}}\right)^T$$



$$C \propto a^r, \quad a \equiv e^{-\kappa}$$

The vectors are not orthogonal! Once they are orthogonalized, their norm gives the ES

For the first two blocks:

$$|\psi_1\rangle = (\alpha_{11}, \alpha_{12}a, \ldots)^T$$

$$\ldots\uparrow\uparrow|\uparrow\uparrow\ldots \quad \ldots\uparrow\uparrow|\downarrow\uparrow\ldots$$

$$|\psi_2\rangle = (\alpha_{21}a, \alpha_{22}a^2, \ldots)^T$$

$$\ldots\uparrow\downarrow|\uparrow\uparrow\ldots \quad \ldots\uparrow\downarrow|\downarrow\uparrow\ldots$$

$$\lambda_1 = \langle\psi_1|\psi_1\rangle \propto \mathcal{O}(1)$$

$$|\psi_2^{(1)}\rangle = |\psi_2\rangle - \frac{\langle\psi_1|\psi_2\rangle}{\langle\psi_1|\psi_1\rangle}|\psi_1\rangle \qquad \lambda_2 = \langle\psi_2^{(1)}|\psi_2^{(1)}\rangle = \mathcal{O}(a^4)$$

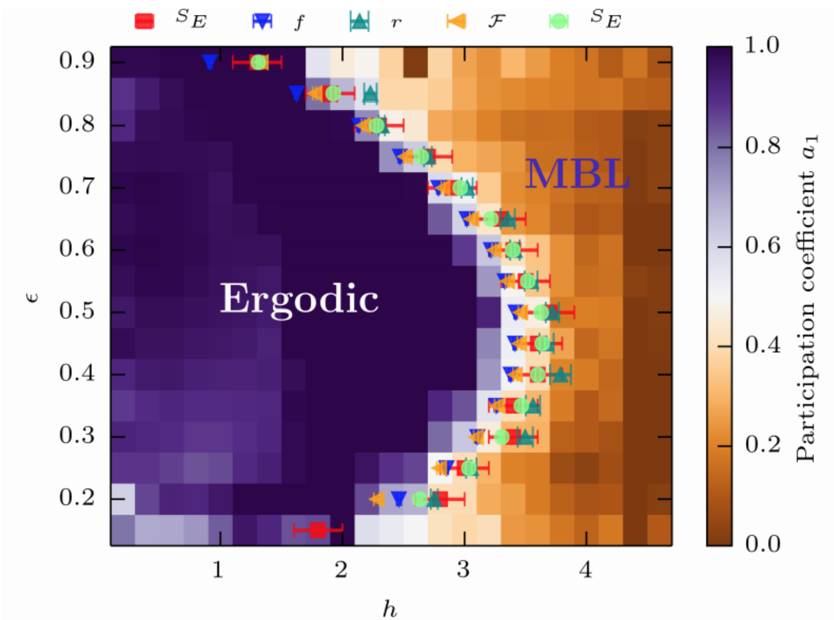
$$\lambda_k^{(r)} = \lambda_{\uparrow\ldots\uparrow\underbrace{\downarrow\ldots\downarrow}_{r}} \propto e^{-4\kappa r} \qquad k = 2^{r-1}+1, \ldots, 2^r$$

$$r \approx \frac{\ln k}{\ln 2} \longrightarrow \lambda_k \propto \frac{1}{k^\gamma} \qquad \gamma = \frac{4\kappa}{\ln 2}$$

[M. Serbyn, A. Michailidis, D. Abanin and ZP, PRL **117**, 160601 (2016)]

Open problems

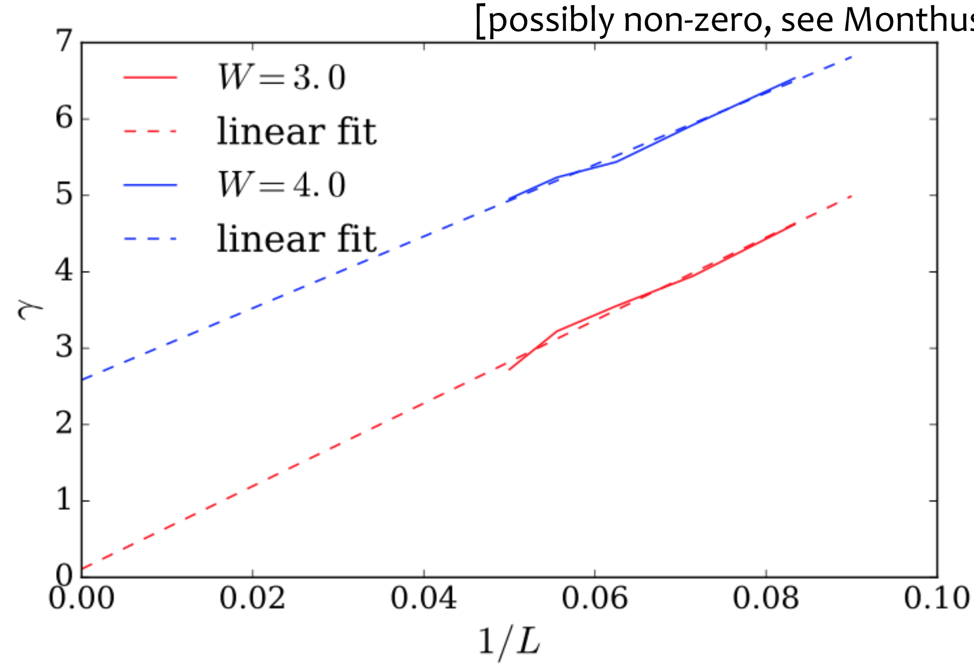
Many-body localization **transition**



[Luitz, Laflorencie, Alet '14]

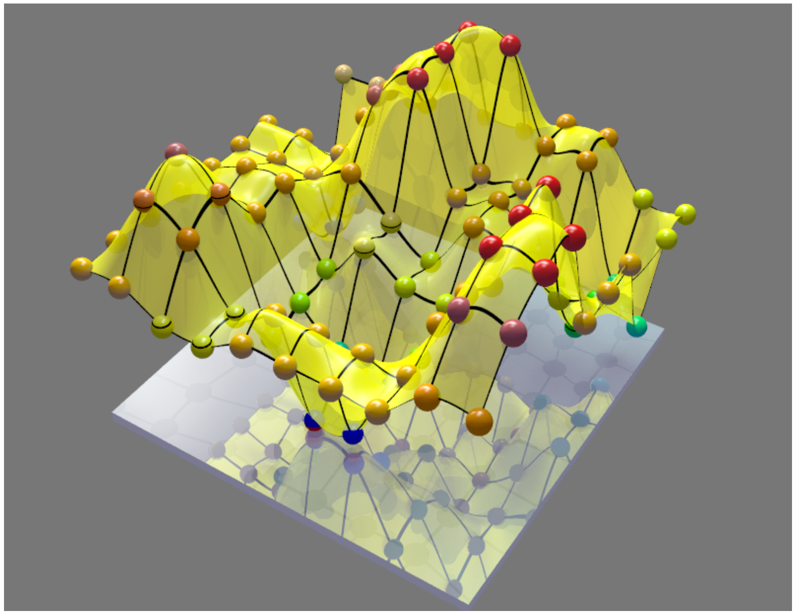
Power-law exponent at the MBL transition?

[possibly non-zero, see Monthus '16]



ES in "glassy" systems **without disorder**

[M. Schiulaz, M. Muller, '13;
T. Grover and M. Fisher, '13;
de Roeck/Huveneers, '13;
Hickey, Genway, Garrahan, '14;
Yao *et al.*, '14;
ZP, Stoudenmire, Abanin '15, ...]



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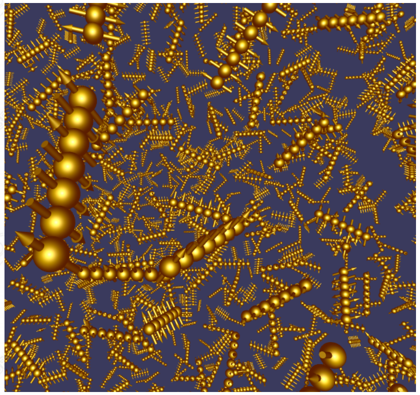
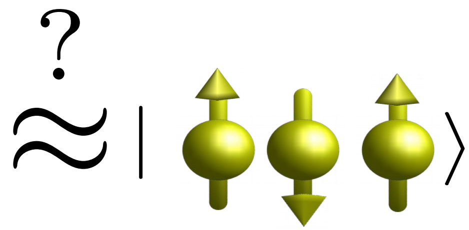
Motivation

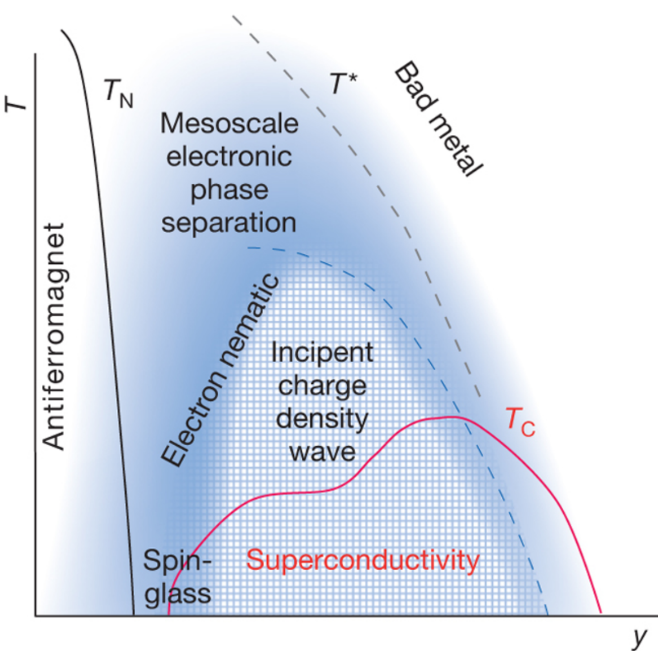




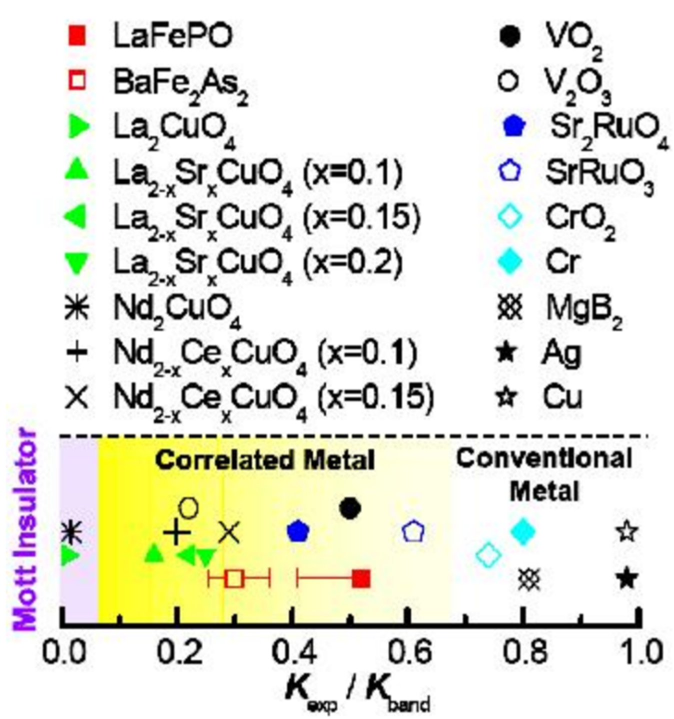
A generic quantum state

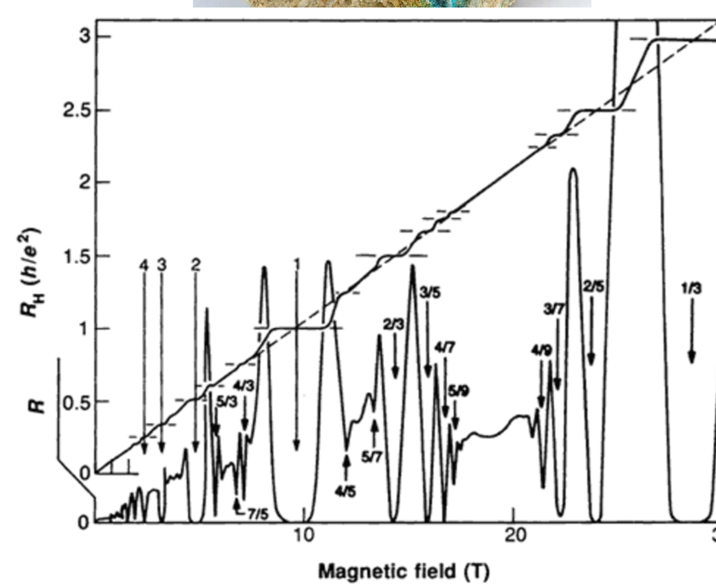
Similar goal to machine learning

TensorFlow

Nearly free systems are easy to understand.
But many interesting systems in nature are not obviously free:

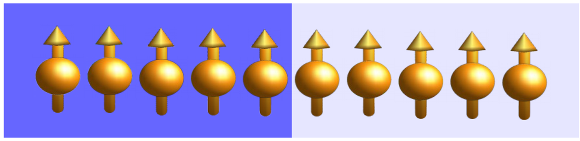






Interaction distance

ρ



Can ρ be distinguished from a free particle density matrix σ?

$$\sigma = e^{z+\sum_j \epsilon_j c_j^\dagger c_j}$$

with some {c} bosonic or fermionic mode operators.

$$D_F(\rho) = \min_{\sigma \in \mathcal{F}} \frac{1}{2} \mathrm{Tr} \sqrt{(\rho - \sigma)^2}$$

F contains all unitary orbits of Gaussian states

Assume ρ, σ have been diagonalized.

We are looking for

$$\min_U \text{trace distance}(\rho, U\sigma U^\dagger)$$

Theorem: minimum (or maximum) is achieved when U is the permutation matrix

[Markham *et al.*, PRA **77**, 042111 (2008)]

Consequence: only need to vary the free entanglement levels $\{\epsilon_1, \ldots, \epsilon_L\}$

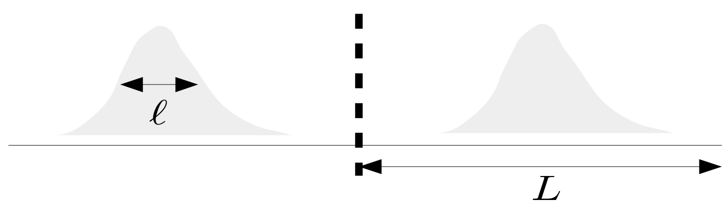
$$E_k^f(\{\epsilon\}) = E_0 + \sum_{i=1}^{L} n_i(k)\epsilon_i$$

$$D_F(\rho) = \min_{\{\epsilon\}} \frac{1}{2} \sum_k |e^{-E_k} - e^{-E_k^f(\{\epsilon\})}|$$

Properties of interaction distance

- Measures **distance** of a given reduced density matrix
- from the manifold of free system's density matrices

- Contains information about **both long-wavelength and short-distance** properties



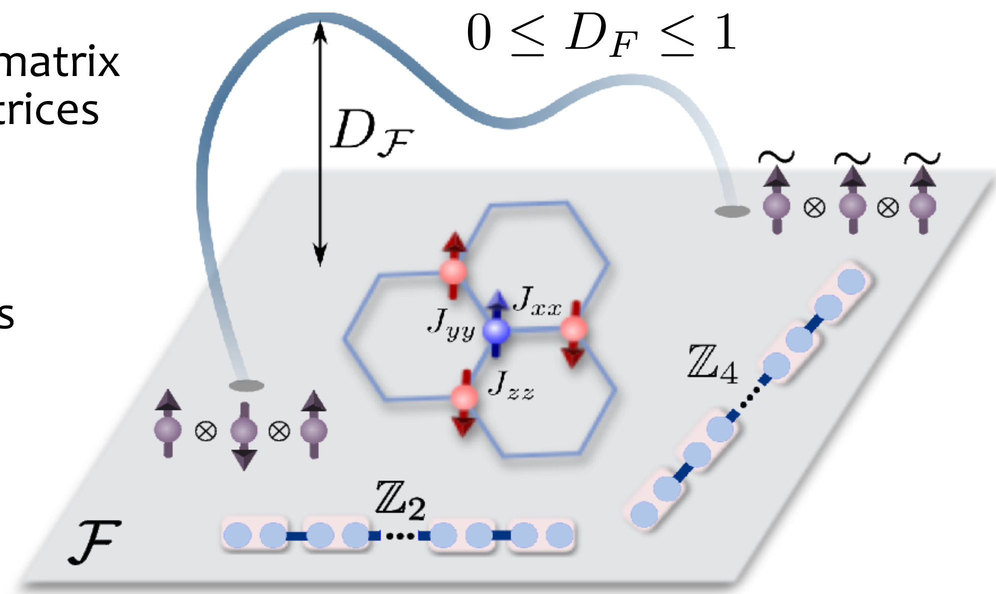
- Generalizes **mean-field theory**; when MF is applicable, then $D_F \to 0$

- Can be calculated **efficiently** if the entanglement spectrum is known $T \sim \mathrm{poly}(\chi)$

- Importantly, the **free quasiparticles are not necessarily of the same statistics** as the original ones

$$|\psi\rangle \simeq U_A \Sigma V_B^{\dagger}$$

We can change the dimensions of entanglement Hilbert space to accommodate e.g., the case where the free quasiparticles in a fermionic system behave as bosons



- Obeys **finite-size scaling** at critical points

$$\boxed{D_F \approx (L^{-1} + \theta)^{\zeta} f((g - g_c)L^{1/\nu})}$$

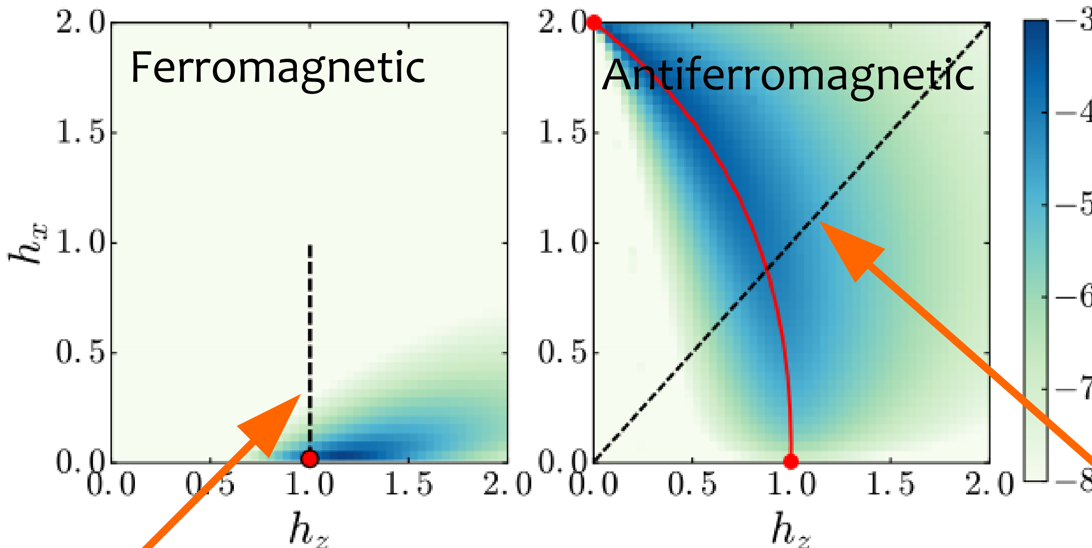
ν = correlation length exponent

ζ = determines whether the interactions are relevant or irrelevant in RG sense

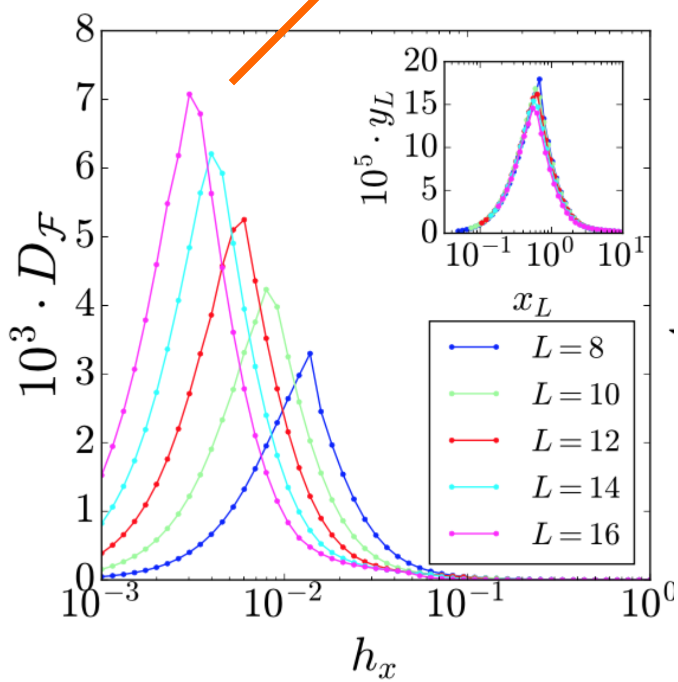
Example: 1D Quantum Ising model

$$H = -\sum_{j=1}^{L} \pm\sigma_j^x\sigma_{j+1}^x - \sum_{j=1}^{L} h_z\sigma_j^z - \sum_{j=1}^{L} h_x\sigma_j^x$$

Longitudinal field (interaction)



AFM phase diagram by DMRG:
Ovchinnikov *et al.*, PRB **68**, 214406 (2003)

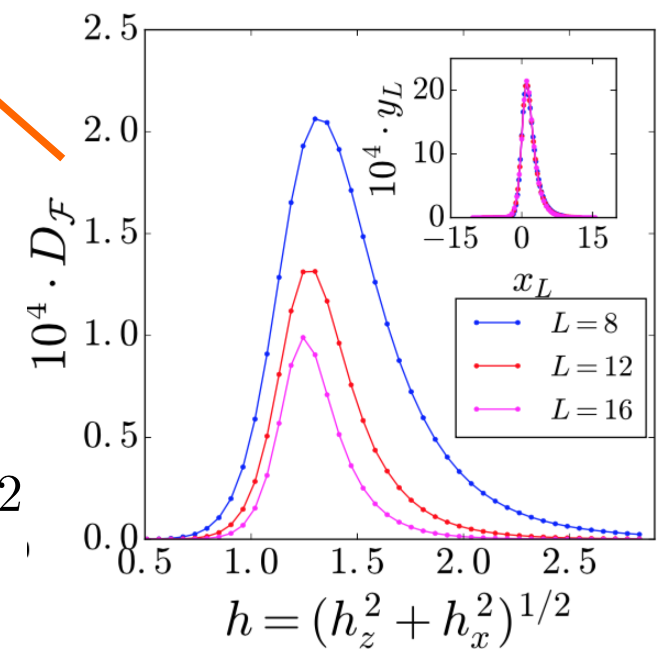


$\nu_{FM} \approx 0.533$

$\zeta_{FM} \approx -1.4$

$\nu_{AFM} \approx 1.252$

$\zeta_{AFM} \approx 1.11$

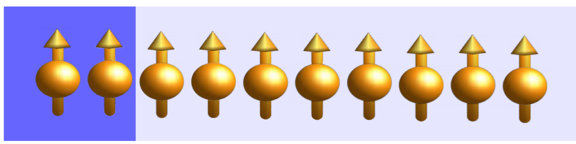


$$\xi \sim |g - g_c|^{-\nu} \quad \nu = \begin{cases} \frac{8}{15}, FM \\ 1, AFM \end{cases}$$

[Zamolodchikov, Int. J. Mod. Phys. A **4**, 4235 (1989)]

Maximally interacting states

ρ



For two fermionic modes,
it can be proven that the state
which maximizes interaction distance is

$$\rho_1 = \rho_2 = \rho_3 = \frac{1}{3}, \rho_4 = 0$$

$$D_F^{max} = \frac{1}{6}$$

Interestingly, this state is the fixed point
of parafermionic Z3 Hamiltonian:

$$H = -\sum_j \tau_j^\dagger \tau_{j+1} + h.c.$$

$$\tau_j = \mathrm{diag}(1, e^{i2\pi/3}, e^{-i2\pi/3})$$

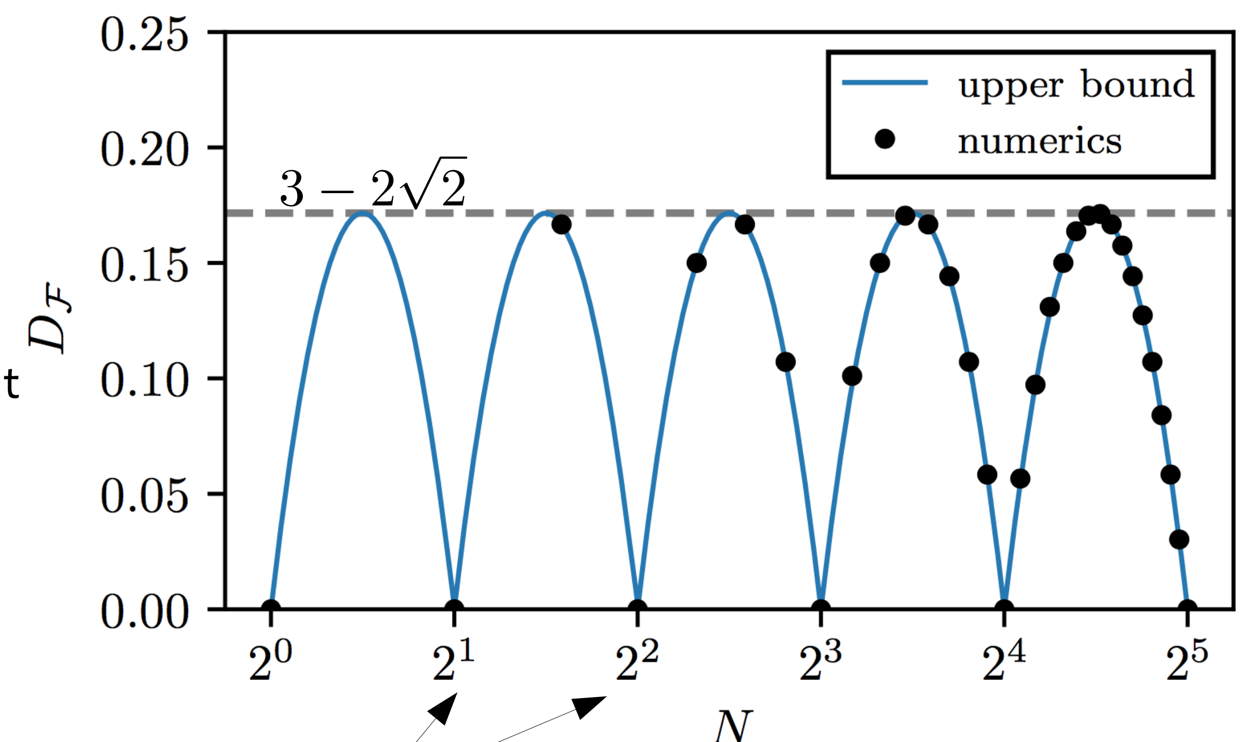
$$\tau_j^3 = 1$$

[see e.g., Jermyn, Mong, Alicea, and Fendley, '14]

We can use the parafermionic ansatz to guess an
upper bound for any number of modes: $D_F \leq 3 - 2\sqrt{2}$

$2^n \leq N < 2^{n+1}$ Guess: $\{\underbrace{\frac{1}{N}, \ldots, \frac{1}{N}}_{N}, 0, 0, \ldots\}$

Match against: $\{\underbrace{\frac{1}{N}, \ldots, \frac{1}{N}}_{2^n}, \underbrace{p, p, \ldots}_{2^n}\}$

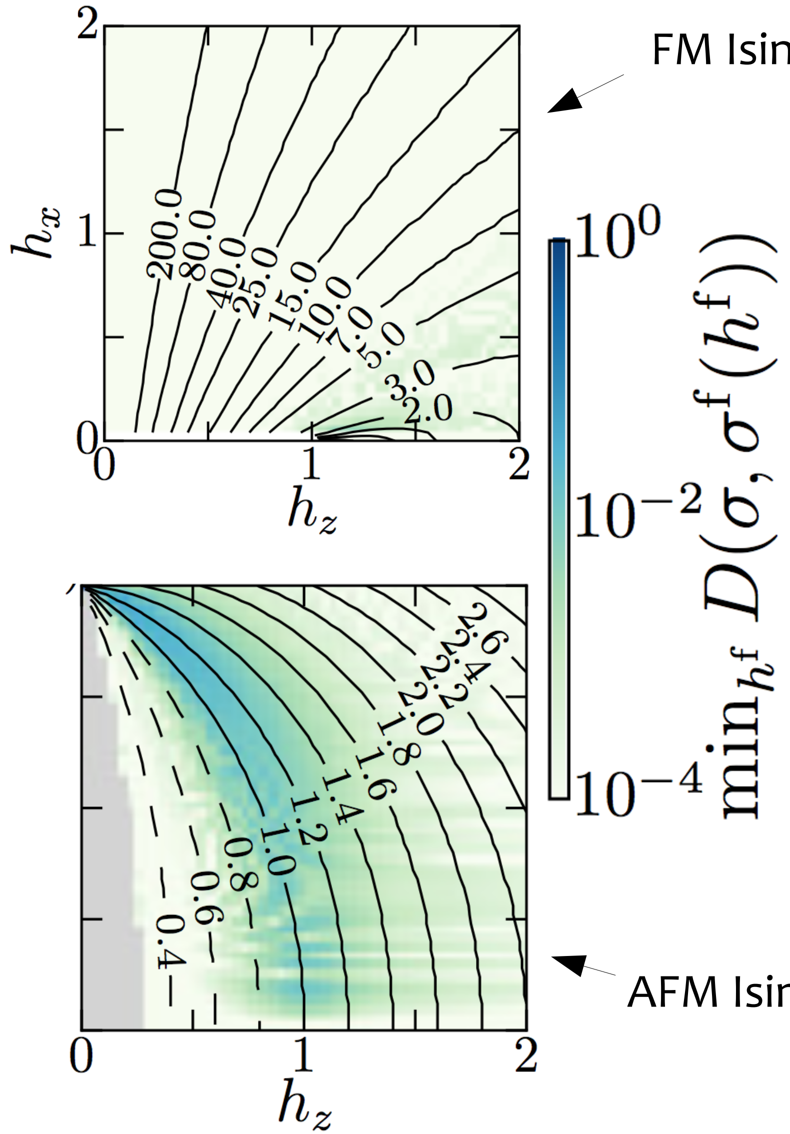


$D_F = 0$ $N = 2^n$

[K. Meichanetzidis, C. Turner, A. Farjami, ZP, J. Pachos, arXiv:1705.09983]

Open questions

(1) Construct an actual closest free model instead of just measuring its distance



(2) Apply this to paradigmatic interacting systems

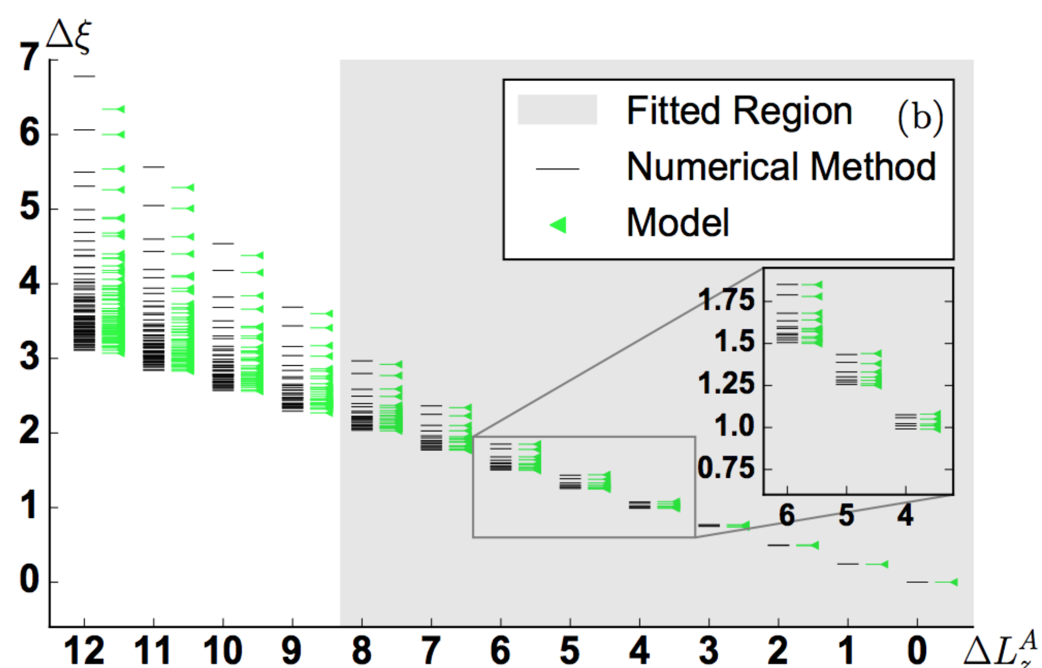
Main candidate: quantum spin liquids, FQHE states...

Integer QH state → FQHE? e.g., 1/3

$$\epsilon_j \approx a_0 + a_1 j + a_2 j^2 + a_3 j^3 + \ldots$$

With few fitting parameters the ES of very large systems can be accurately modeled

[Davenport *et al.*, PRB **92**, 115155 (2015)]

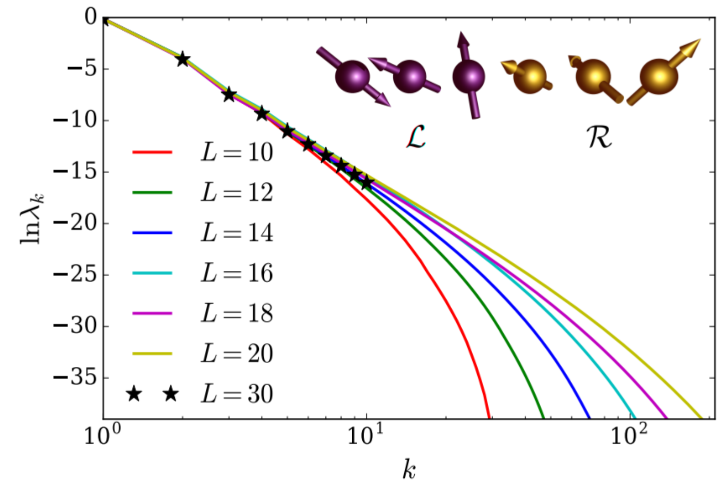


[Rodriguez and Sierra '09; Sterdyniak *et al.* '12; Dubail, Read, Rezayi '12]

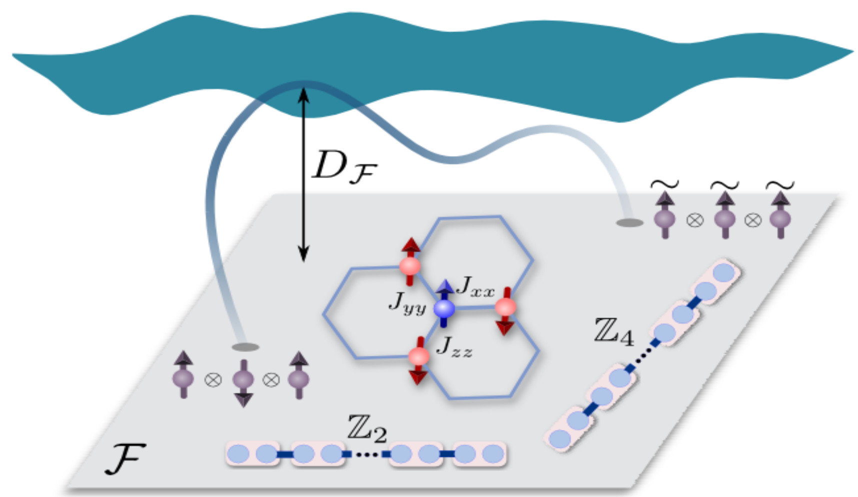
Conclusions

- Entanglement spectrum still reveals new aspects of many-body systems
- In strongly disordered systems,
  the ES has a universal power-law structure

  (a consequence of local integrals of motion)

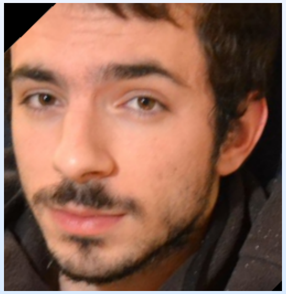


- "Interaction distance"
  measures how far a many-body
  state is from the closest free state.



Potentially useful for identifying "most interacting" points in the phase diagrams of quantum many-body system, where new physics may be hiding.

Acknowledgments



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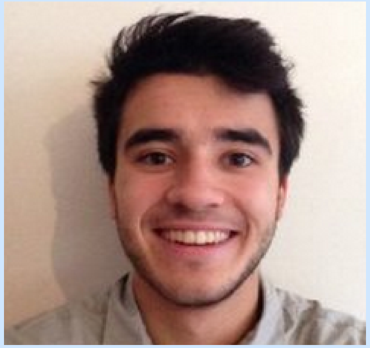

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