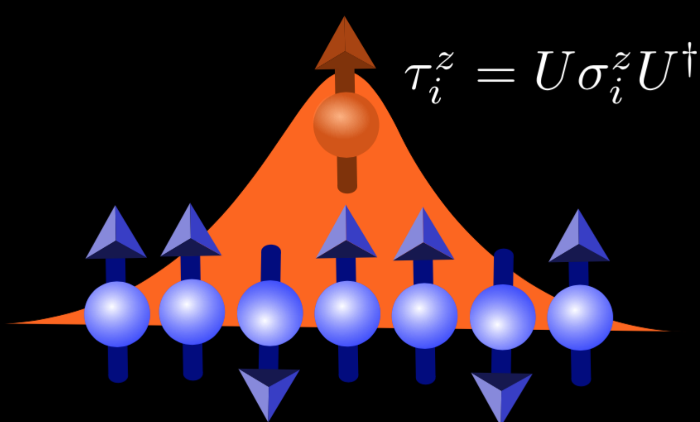
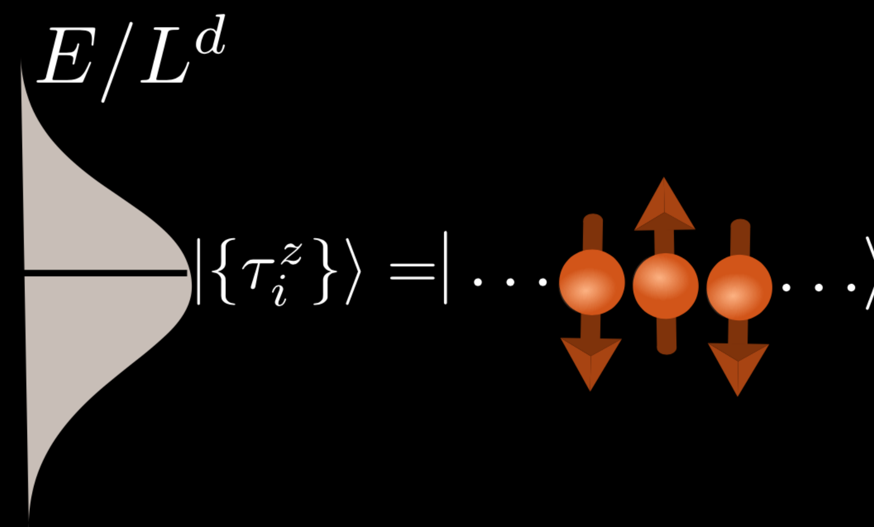
# Introduction to many-body localization

Zlatko Papic




"Localization in Quantum Systems", King's College London, 2/6/2017

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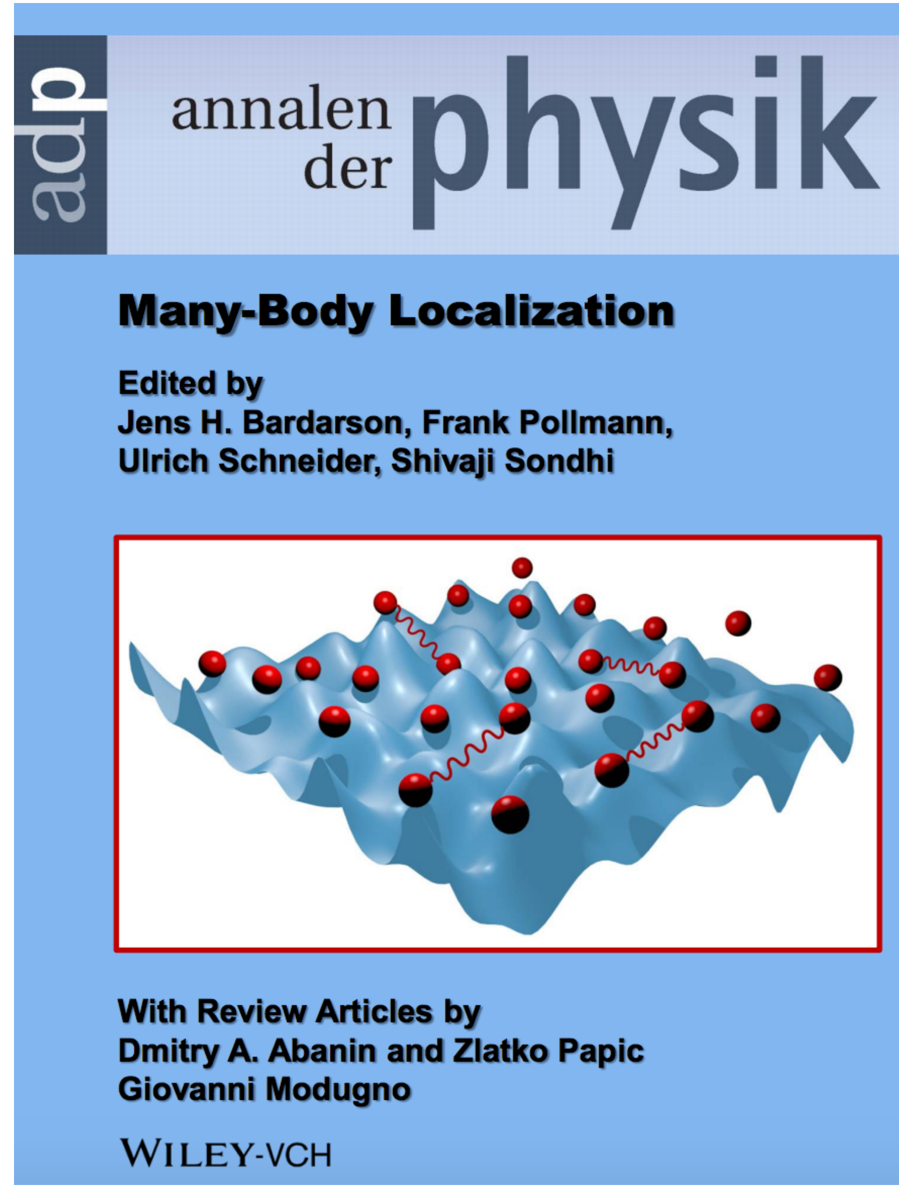
# Plan

- Quantum many-body problem

- Ergodic systems
  and eigenstate thermalization
  hypothesis

- Many-body localization:
  Local integrals of motion

- Experiments and open problems



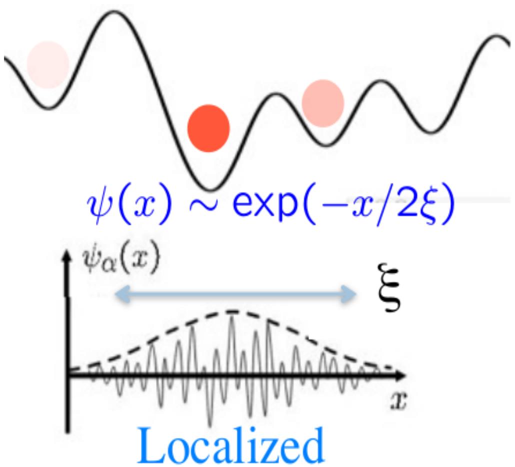

[Abanin & Papic, arxiv:1705.09103]

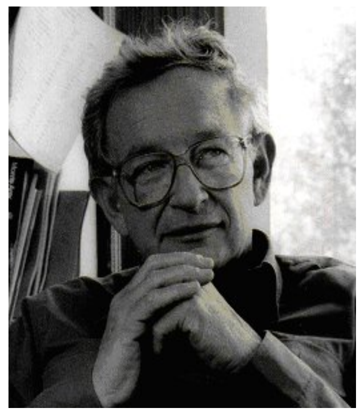
Another recommended review:
[Nandkishore & Huse, arXiv:1404.0686]

# Motivation

Narrow:

Understand the stability of Anderson insulator when perturbed by interaction between particles (finite density of particles, finite interaction strength)



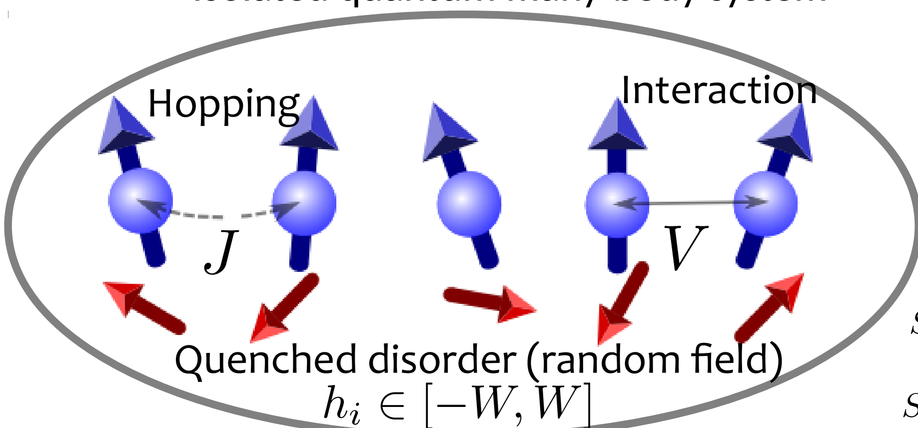



Broader:

**What are the generic behaviors of isolated quantum systems at arbitrary energy density? (open problem even in 1D)**



Isolated quantum many-body system




Jordan-Wigner

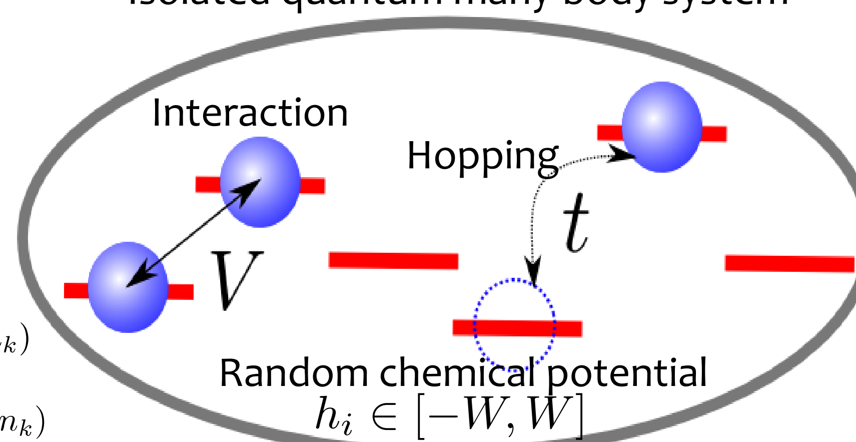
$$\approx$$

$$S_i^z = c_i^\dagger c_i - 1/2$$

$$S_i^+ = c_i^\dagger \exp(i\pi \sum_{k<i} n_k)$$

$$S_i^- = c_i \exp(-i\pi \sum_{k<i} n_k)$$

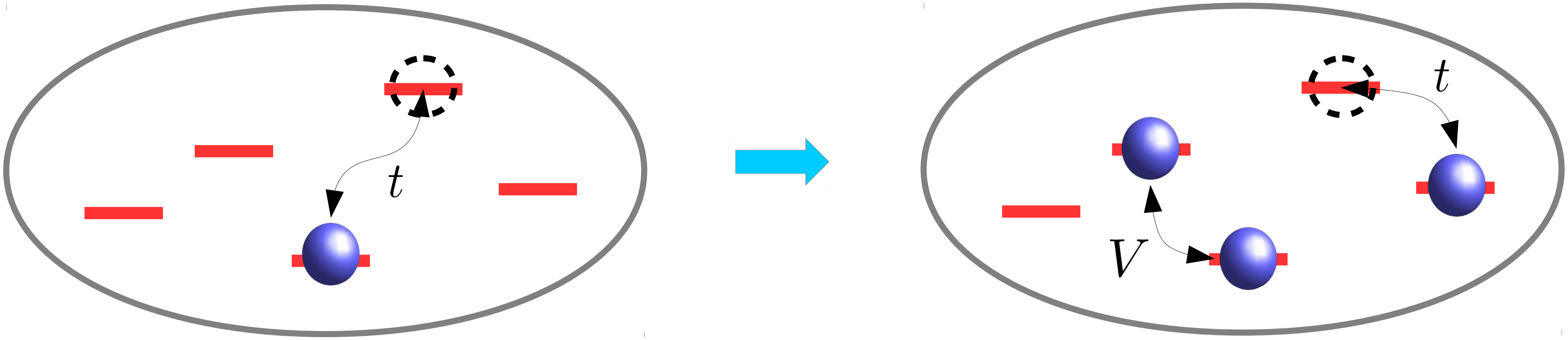
Isolated quantum many-body system




$$H = J\sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + V\sum_i \sigma_i^z \sigma_{i+1}^z + \sum_i h_i \sigma_i^z$$

$$H = J\sum_i c_i^\dagger c_{i+1} + h.c. + V\sum_i \rho_i \rho_{i+1} + \sum_i h_i \rho_i$$

# One body vs. many body



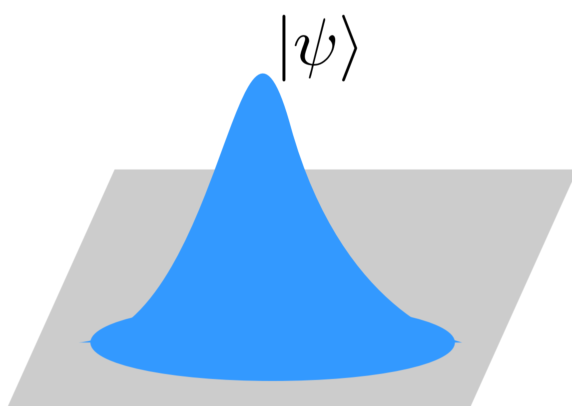

$$H = J\sum_i c_i^\dagger c_{i+1} + h.c. + \sum_i h_i \rho_i$$

$$H = J\sum_i c_i^\dagger c_{i+1} + h.c. + V\sum_i \rho_i \rho_{i+1} + \sum_i h_i \rho_i$$

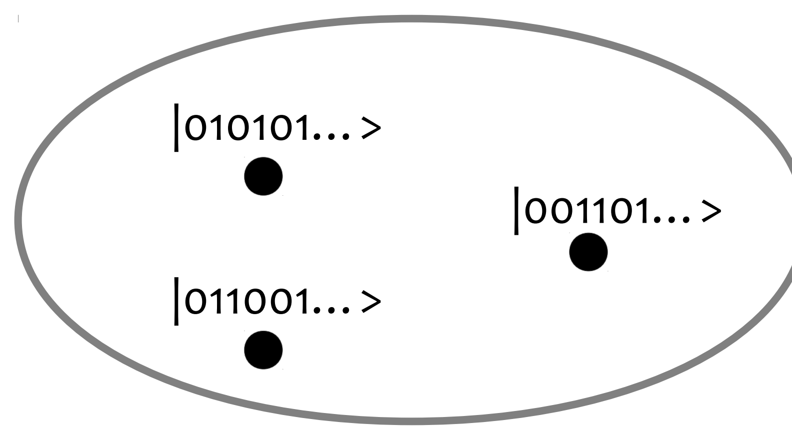
Complexity increases linearly with the number of lattice sites $\sim L$

Complexity increases exponentially with the number of lattice sites $\sim 2^L$

Wavefunction has a direct real space interpretation




Wavefunction lives in Fock space; no direct real space interpretation



# Quantum entanglement

$|0\rangle_A \otimes |1\rangle_B$

product state

$$\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$$

entangled state = cannot be written as a product state in any basis

$$|0\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B$$

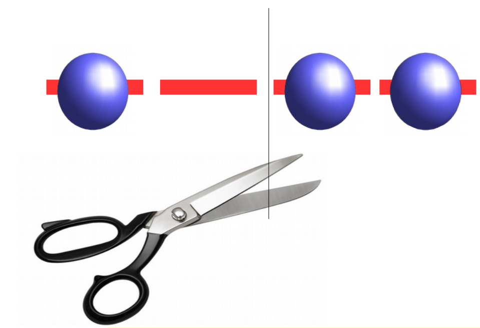
entangled?

$$= (|0\rangle_A + |1\rangle_A) \otimes (|0\rangle_B + |1\rangle_B)$$ **NO!**

How to quantify entanglement?

Reduced density matrix:

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$



**Schmidt decomposition (SVD)**

$$|\psi\rangle = \sum_{a,b} C_{ab}|a\rangle \otimes |b\rangle$$

$$|\psi\rangle \to C = A\Lambda B^\dagger$$

$$|\psi\rangle = \sum_k \lambda_k |a_k\rangle|b_k\rangle$$

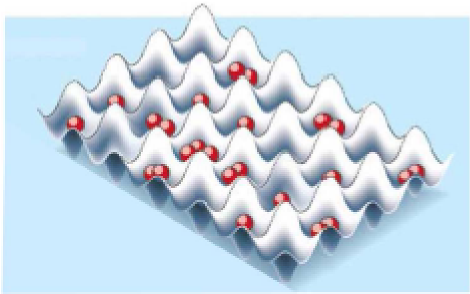
$$S_A = -\sum_k \lambda_k^2 \ln \lambda_k^2$$

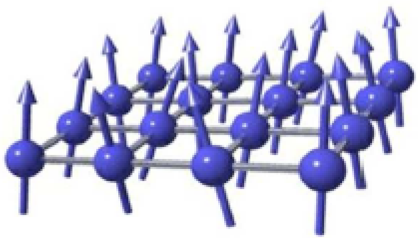
**Entanglement entropy** $$S_A = -\mathrm{tr}_A \rho_A \ln \rho_A$$

| | |
|---|---|
| $S_A = 0$ | Unentangled (product) state |
| $S_A \propto \mathrm{vol}_A$ | Thermal states<br>Random states $S_A \approx \ln(d_A) - \frac{d_A}{2d_B}$ |
| $S_A \propto \mathrm{area}_A$ | Gapped, ordered states |
| $S_A \propto \mathrm{area}_A \times \ln L$ | Gapless states, Fermi surfaces |

# Dynamics of entanglement

Probe inspired by cold atom systems





Highly tunable (Hamiltonian and state engineering)
Almost totally isolated (closed systems)
Long natural timescales (kHz vs. GHz -THz in solids)

1. Prepare a well-defined (simple) initial state

$$|\psi_0\rangle = |\ldots \uparrow\downarrow\uparrow\downarrow \ldots\rangle$$

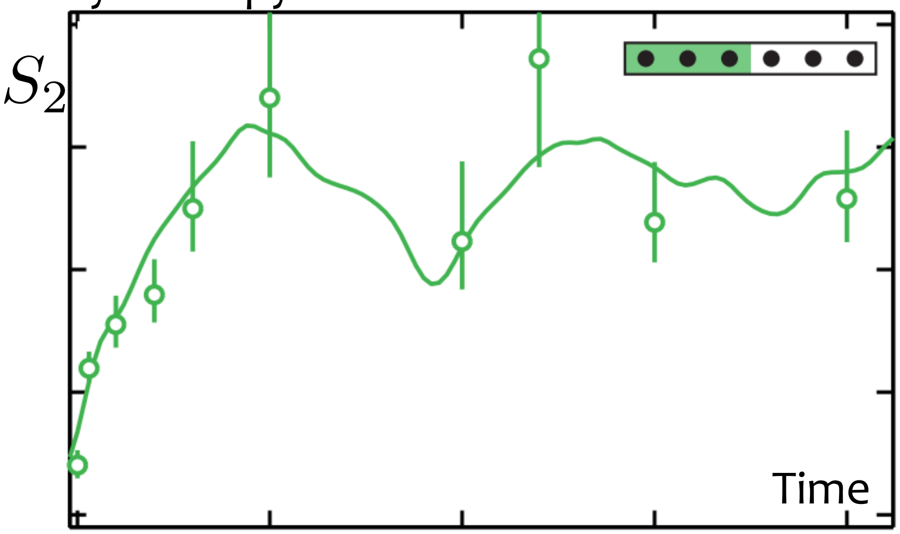
2. Evolve with a known Hamiltonian

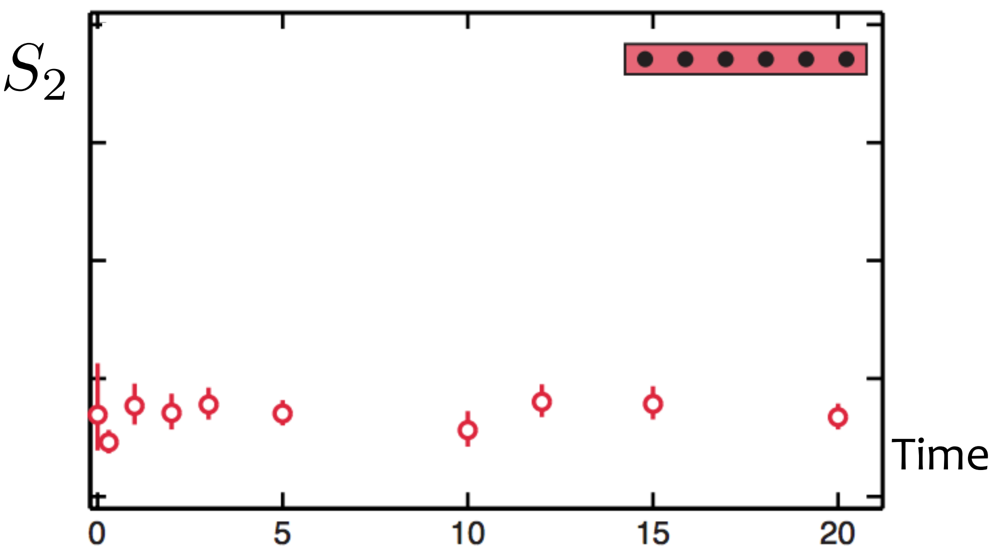
$$|\psi(t)\rangle = e^{-itH}|\psi_0\rangle$$

3. Observe at later times

Rb atoms in a 2D optical lattice (Bose-Hubbard model) [A. M. Kaufman *et al*, arXiv:1603.04409v3]
[see also C. Neill *et al*, arXiv:1601.00600]

Renyi entropy




As the system evolves, entanglement spreads/information distributed into non-local observables

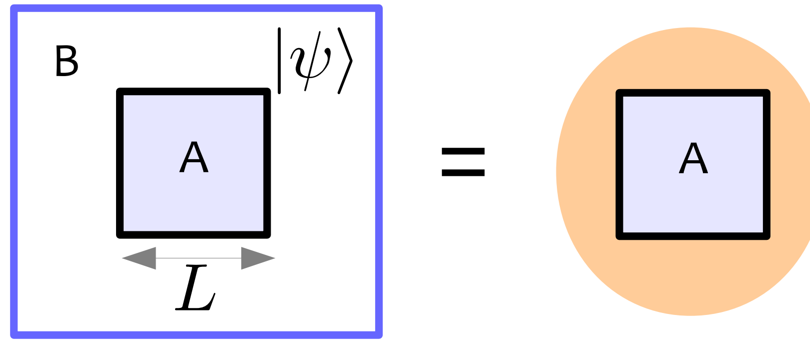
# Eigenstate Thermalization Hypothesis

Dynamics of a system
is determined by the properties of
its eigenstates

[Deutsch, PRA 43, 2146 (1991);
Srednicki, PRE 50, 888 (1994)]

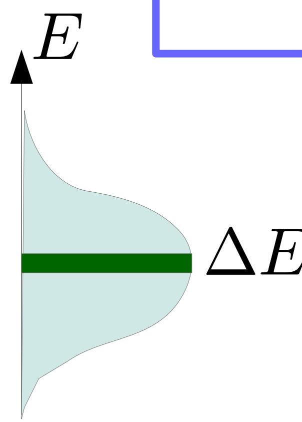
$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$
$$= \sum_\alpha e^{-i\epsilon_\alpha t}|\epsilon_\alpha\rangle\langle\epsilon_\alpha|\psi(0)\rangle$$

$$H|\psi\rangle = E|\psi\rangle$$




Thermalizing system =
individual eigenstates look thermal

Highly excited eigenstates =
resemble random vectors




$$\rho_A = \mathrm{tr}_B|\psi\rangle\langle\psi| \approx \rho_{th} = \frac{1}{Z}e^{-\beta H}$$

ETH also makes predictions
about dynamics:

$$\overline{(O_t - \bar{O})^2} = \mathcal{O}(e^{-S})$$

Infinite time average

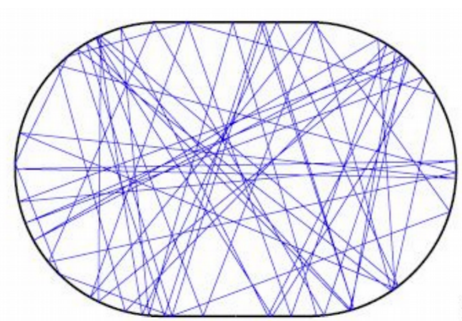
$$\langle\hat{O}\rangle_E = \langle\hat{O}\rangle_{th} = \langle\hat{O}\rangle_{E+\Delta E}$$

$$S_{ent}(A) = S_{th}(A) \sim L^d$$

Depends on
properties
of states

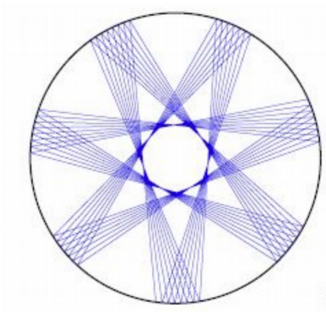
From the counting
of states
at temperature T

ETH systems are **ergodic**
= they explore all configurations
allowed by global conservation laws



Random matrix theory
[see V. Kravtsov, arxiv:0911.0639]

# Systems that do not thermalize: Anderson localization



Non-ergodic classical system

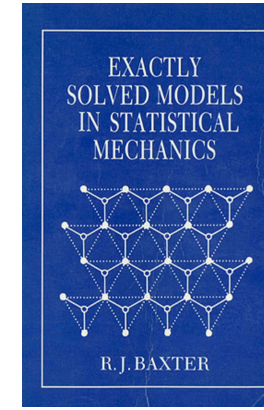
KAM theorem

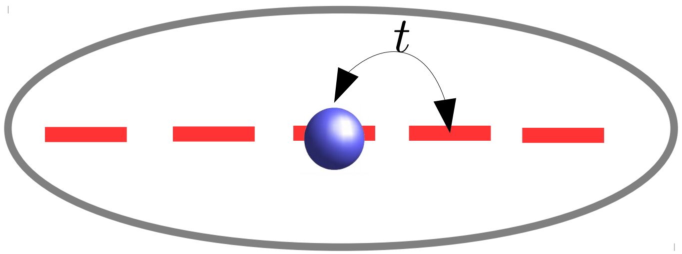
Quantum integrable systems

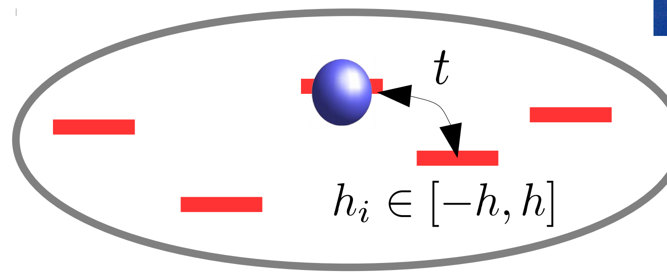
$$[H, K_i] = 0 \qquad [K_i, K_j] = 0$$

Ex: Lieb-Liniger, Hubbard, XXZ

**Non-generic**

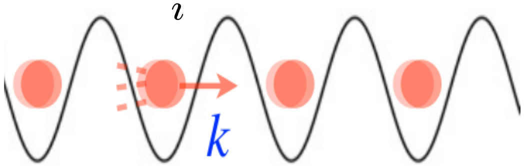


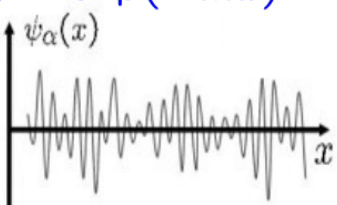




$$H = J \sum_i c_i^\dagger c_{i+1} + h.c.$$

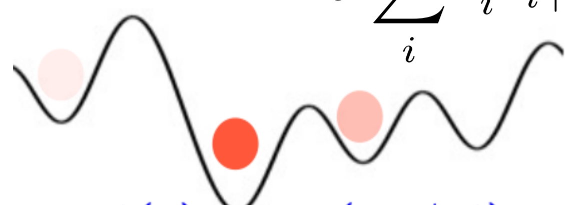
$$H = J \sum_i c_i^\dagger c_{i+1} + h.c. + \sum_i h_i \rho_i$$



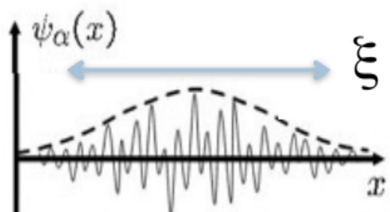

$\psi(x) \sim \exp(-ikx)$



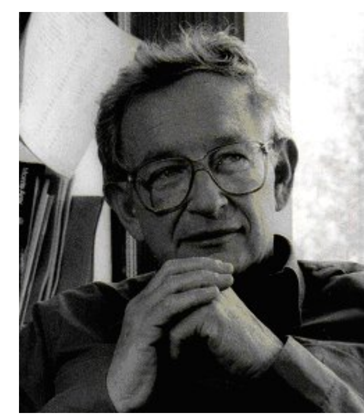

Extended



$\psi(x) \sim \exp(-x/2\xi)$




Localized



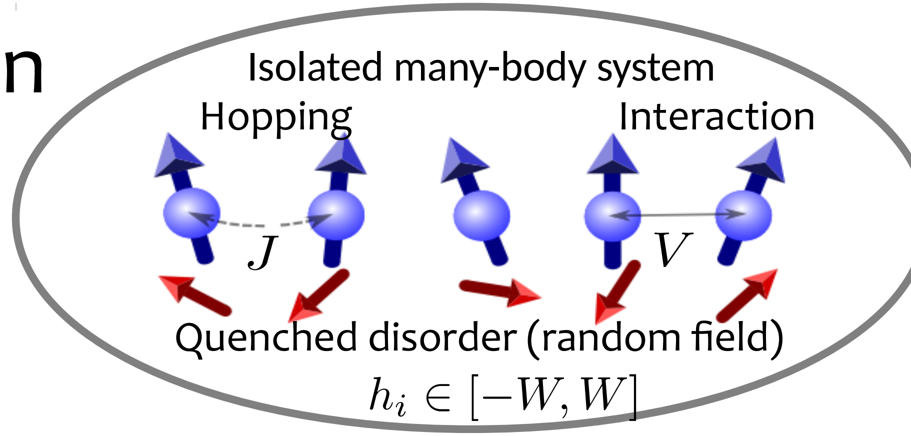
**Generic**

Single-particle states localized with localization length $\xi$

States close in energy, far apart In space (no overlap)

# Many-body localization

$$H = J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + V\sum_i \sigma_i^z\sigma_{i+1}^z + \sum_i h_i\sigma_i^z$$




$$J = V = 1$$

- An infinite system is a heat bath
- System achieves thermal equilibrium
- Finite D.C. transport
- Obeys ETH
- Extensive entanglement $\quad S_{ent} \propto L^d$
- Fast spreading of entanglement

$$S_{ent}(t) \sim t^{\alpha}$$

Anderson insulator (infinite disorder)

Integrable

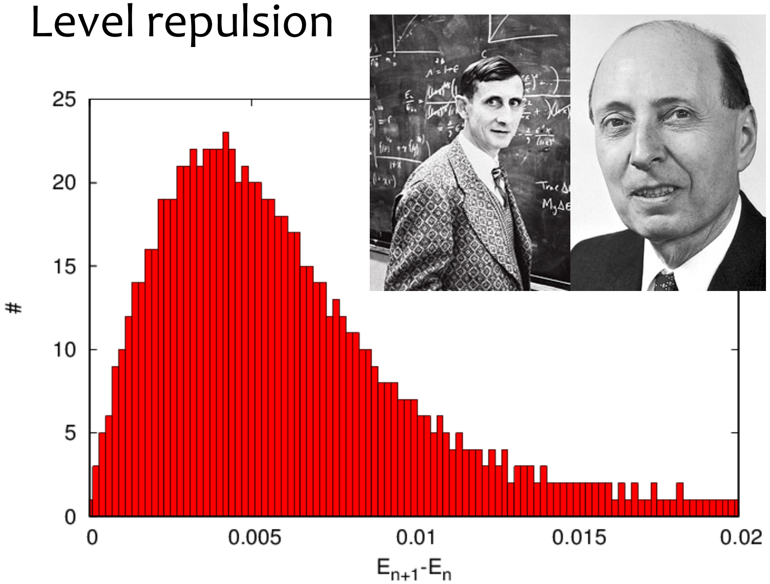
0

Thermal phase

**“Many-body localized” phase**

Disorder strength

Level repulsion



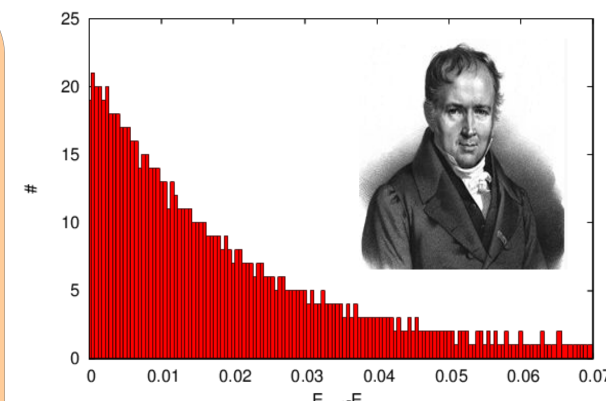

- An infinite system is not a heat bath
- Doesn’t thermally equilibrate
- ETH false
- D.C. transport is zero
- “Area-law” entanglement

$$S_{ent} \propto L^{d-1}$$

- Slow growth of entanglement in quench

$$S_{ent}(t) \sim \ln(t)$$

Poisson level statistics



# Level statistics

## Level repulsion – avoided crossing:

Example: two levels

$$H = \begin{pmatrix} \epsilon_1 & V/\sqrt{2} \\ V/\sqrt{2} & \epsilon_2 \end{pmatrix}$$

$$P(\epsilon_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2\sigma^2}} \quad \text{etc.}$$

$$\downarrow$$

$$E_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \pm \frac{1}{2}\sqrt{(\epsilon_1 - \epsilon_2)^2 + 2V^2}$$

$$P(E_1 - E_2 = \omega) =$$

$$= \int d\epsilon_1 d\epsilon_2 dV P(\epsilon_1) P(\epsilon_2) P(V) \delta(E_1 - E_2 - \omega)$$

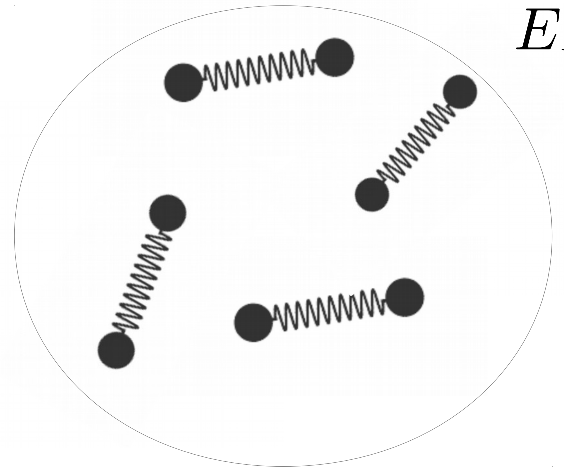
$$= \ldots = \frac{\omega}{2\sigma^2} e^{-\frac{\omega^2}{4\sigma^2}}$$

**Wigner surmise:**

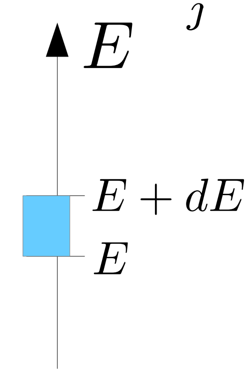
$$P(\omega) = A_\beta w^\beta e^{-B_\beta w^2}$$

specifies the ensemble

## Poisson statistics – uncorrelated levels



$$E_{\{n_j\}} = \sum_j n_j \epsilon_j$$

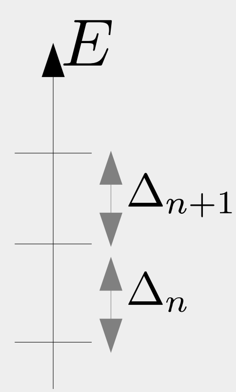



$$P_n = \frac{\lambda^n}{n!} e^{-\lambda}$$

average number of levels in interval

The probability that there are no levels is **finite**.

### Characterising distributions with one parameter




$$r = \frac{\min(\Delta_n, \Delta_{n+1})}{\max(\Delta_n, \Delta_{n+1})}$$

$\langle r \rangle \approx 0.53$ GOE

$\langle r \rangle \approx 0.39$ Poisson

# Some landmarks in the history of MBL

- 1958: Anderson introduced the MBL problem but ended up simplifying it down to one particle
  [P. W. Anderson Phys. Rev. 109, 1492 (1958)]

- Several works attempted to put interactions back during the 1970s, 1980s but status remained unclear; popular belief was that interactions restore transport

- 2006: Basko, Aleiner & Altshuler (BAA) perturbative treatment: localization survives for finite range of interactions [D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Annals of Physics 321, 1126 (2006)]

- 2010: Pal & Huse first numerical studies on lattice models, roughly confirming BAA
  [A. Pal and D. Huse, PRB 82, 174411 (2010)]

- 2012: More numerics from Bardarson, Pollmann & Moore confirm a logarithmic growth of entanglement in MBL systems found in 2008 by Znidaric, Prosen & Prelovsek
  [M. Znidaric, T. Prosen, and P. Prelovsek PRB 77, 064426 (2008); J. Bardarson, F. Pollmann, and J. Moore, PRL 109, 017202 (2012)]

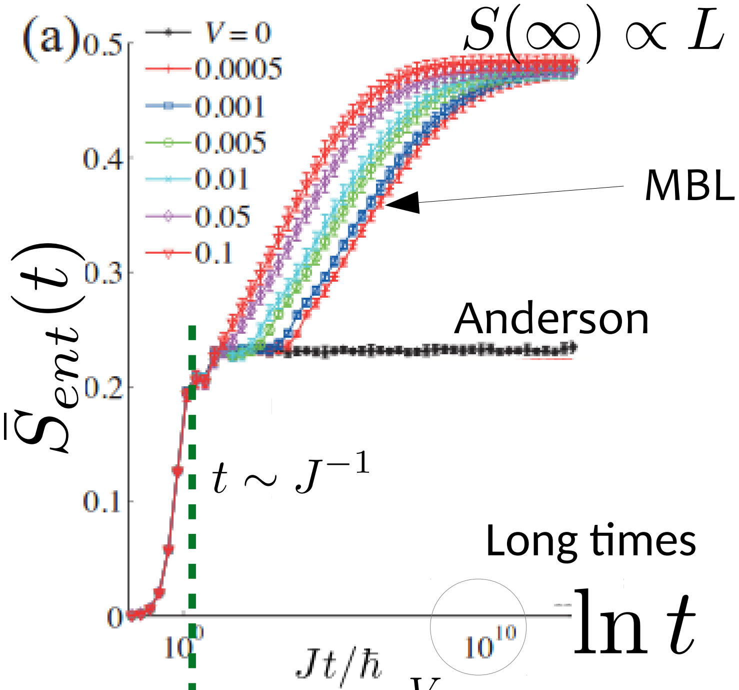
- 2013: Phenomenological picture of local integrals of motion emerges
  [M. Serbyn, Z. Papicc, and D. A. Abanin, PRL111, 127201 (2013;. D. A. Huse, R. Nandkishore, and V. Oganesyan, PRB 90, 174202 (2014)]

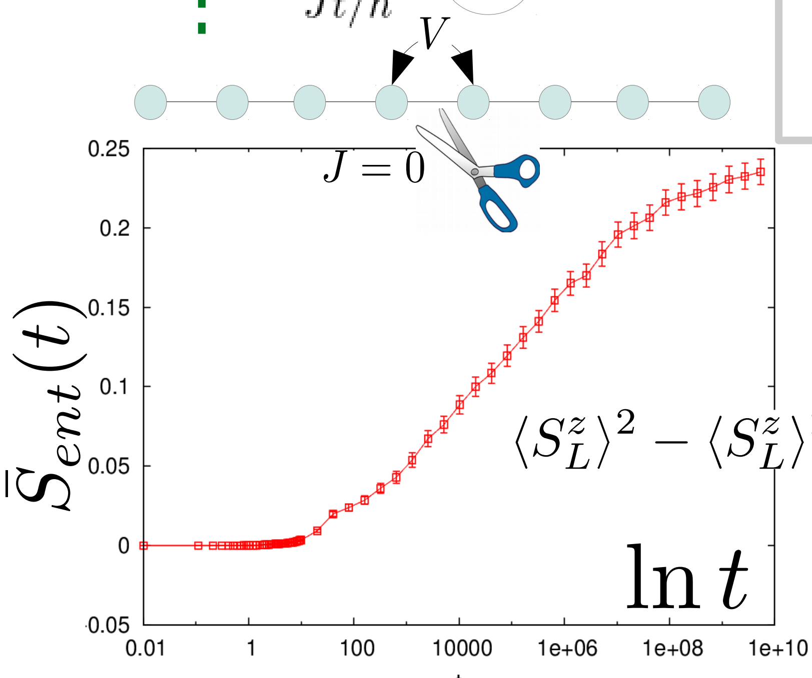
- 2014: Proof by Imbrie that LIOMs can be rigorously defined in a specific model
  [J. Z. Imbrie, Journal of Statistical Physics 163 (5), 998–1048 (2016).]

- 2015/16: First experiments looking for MBL in cold atoms and trapped ions
  [M. Schreiber *et al.*, Science 349 (6250), 842–845 (2015); J. Smith *et al.*, Nature Physics 12, 907–911 (2016)]

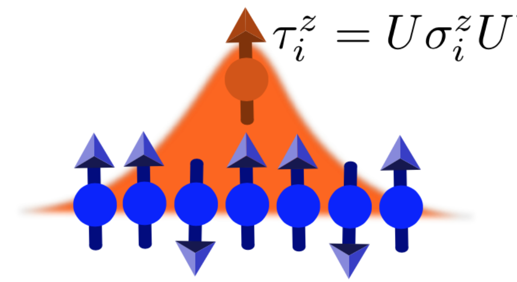
# Local integrals of motion

A puzzling result in 2012:
[Znidaric, Prosen, Prelovsek ‘08; Bardarson, Pollmann, Moore, ‘12]




$$\langle S_L^z \rangle^2 - \langle S_L^z \rangle^2 \approx 0$$




(1) z-projections of spins are approximately conserved = emergence of pseudospins $\tau_i^z$

$$\tau_i^z \approx \sigma_i^z + \sum_{j,k} \sum_{a,b=x,y,z} f_{i;jk}^{ab} \sigma_j^a \sigma_k^b + \dots$$

$$f_{i;jk} \propto \exp(-\max\{|i-j|, |i-k|\}/\xi)$$

“many-body localization length”

(2) any two subsystems get entangled but exponentially slow – there is a direct effective interaction between pseudospins $J_{ij} \propto e^{-|i-j|/\tilde{\xi}}$

Phenomenological Hamiltonian in the MBL phase:

$$H = \sum_i h_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \dots$$

Local integrals of motion:

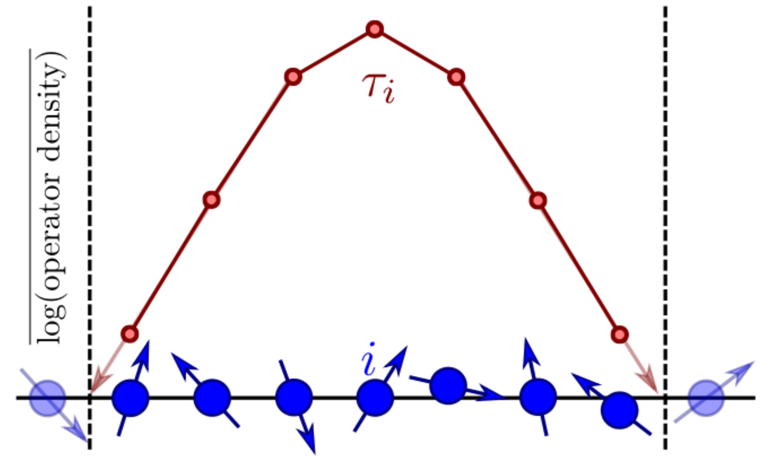
$$[\tau_i^z, H] = 0, \; [\tau_i^z, \tau_j^z] = 0$$

[Serbyn, ZP, Abanin, '13; Huse, Nandkishore, Oganesyan, '13; Imbrie '14; Chandran *et al.* '14; Ros *et al.*, '14]

# Local integrals of motion – explicit construction

Various approaches:

* Time averaging [Chandran et al, '14]
* Rigorous proof near classical limit [Imbrie, '14]
* Perturbative à la Basko *et al* [Ros et al, '14]
* Similarity transform [Rademaker, Ortuno '16]
* "Spectrum bifurcation" RG [You et al, '16]
* QMC [Inglis, Pollet '16]

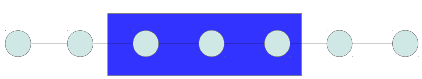



[T. O'Brien, D. Abanin, G. Vidal, ZP, arXiv:1608.03296]

Direct minimization

[H. Kim, M. C. Banuls, J. I. Cirac, M. B. Hastings, and D. A. Huse, '15]

$$A = \sum_i a_i A_i$$



$$\min_A \frac{\mathrm{Tr}\left([A,\mathcal{H}]^\dagger [A,\mathcal{H}]\right)}{\mathrm{Tr} A^\dagger A}$$

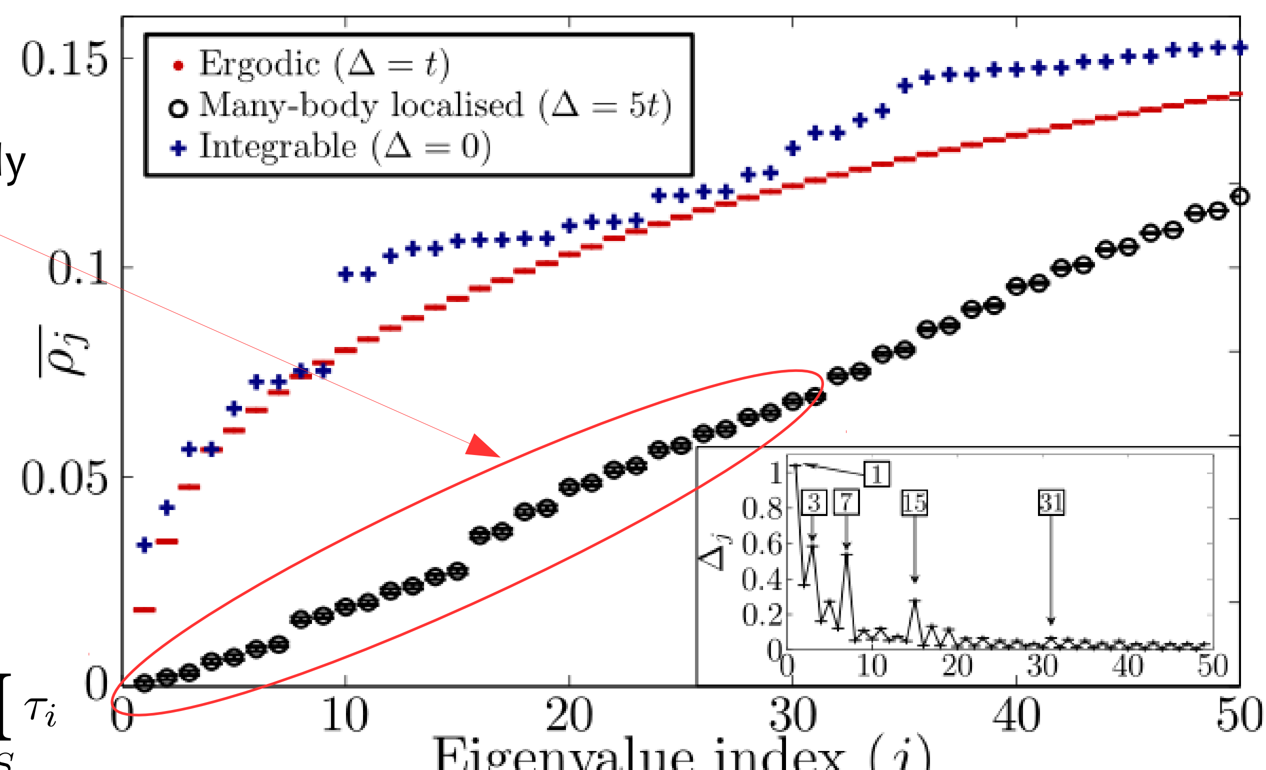
$$\mathcal{C}_{ij} \equiv \mathrm{Tr}\left([A_i,\mathcal{H}]^\dagger [A_j,\mathcal{H}]\right)$$

"localization matrix"

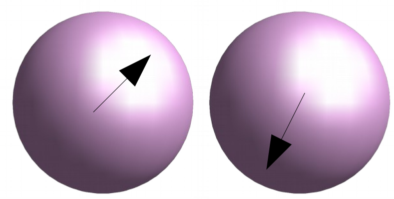
Extensively many MBL operators

$$P_S = \prod_{i \in S} \tau_i$$

Localization area = 7 sites (5 in bulk)



# Consequence of Local Integrals of Motion: Quench Dynamics



Prepare initial product state of effective spins:

$$|\psi(0)\rangle = A_{1\uparrow}A_{2\uparrow}|\uparrow\uparrow\rangle + A_{1\uparrow}A_{2\downarrow}|\uparrow\downarrow\rangle + A_{1\downarrow}A_{2\uparrow}|\downarrow\uparrow\rangle + A_{1\downarrow}A_{2\downarrow}|\downarrow\downarrow\rangle$$

Time evolve:

$$|\psi(t)\rangle = A_{1\uparrow}A_{2\uparrow}e^{-itE_{\uparrow\uparrow}}|\uparrow\uparrow\rangle + A_{1\uparrow}A_{2\downarrow}e^{-itE_{\uparrow\downarrow}}|\uparrow\downarrow\rangle + A_{1\downarrow}A_{2\uparrow}e^{-itE_{\downarrow\uparrow}}|\downarrow\uparrow\rangle + A_{1\downarrow}A_{2\downarrow}e^{-itE_{\downarrow\downarrow}}|\downarrow\downarrow\rangle$$

$$\rho_{\mathrm{off-diag}}(t) = A_{1\uparrow}A_{1\downarrow}[A^2_{2\uparrow}e^{-it(E_{\uparrow\uparrow}-E_{\downarrow\uparrow})} + A^2_{2\downarrow}e^{-it(E_{\uparrow\downarrow}-E_{\downarrow\downarrow})}]$$

For one spin out of many spins: $\rho_{\mathrm{off-diag}}(t) = A_{1\uparrow}A_{1\downarrow}\sum_{\{\tau'\}}\prod_{i>1}A^2_{i\{\tau'\}}e^{-it(\boxed{E_{\uparrow\{\tau'\}}-E_{\downarrow\{\tau'\}}})}$

$2\tilde{H}_1$

Effective magnetic field for spin 1 (random)

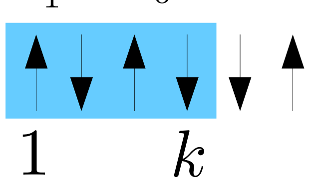
$$H = \sum_i \tau_i^z \left(\boxed{h_i + \sum_j J_{ij}\tau_j^z + \ldots}\right)$$

For spin no.1: $\tilde{H}_1 = \sum_k \tilde{h}_1^k$

Sum of random numbers, but at a given time only a finite number of terms contribute: $k \sim \xi \ln(J_0 t)$

$$\rho_{\mathrm{off-diag}}(t \gg 1/J_0) \sim \frac{A_{1\uparrow}A_{1\downarrow}}{\sqrt{2^k}} \propto \frac{1}{(J_0 t)^a}$$

$\tilde{h}_1^k \sim J_0 e^{-k/\xi}$




Final result:

$$\boxed{\begin{aligned}\langle\tau_1^z(t)\rangle &= \langle\tau_1^z(0)\rangle \\ \langle\tau_1^{x,y}(t)\rangle &\propto \frac{1}{t^a}\end{aligned}}$$

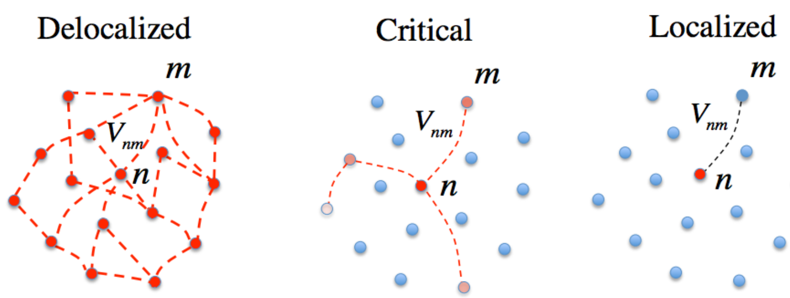
Also holds for more general initial states, as well as realistic initial states (i.e., product states of physical spins)

[Serbyn, Papic, Abanin, ‘14]

# Matrix elements of local operators

Can a local perturbation hybridize the eigenstates?

$$\langle \mathcal{G}(\epsilon, L)\rangle = \langle \ln \frac{\langle n|\mathcal{O}|n+1\rangle}{E_{n+1}-E_n}\rangle$$



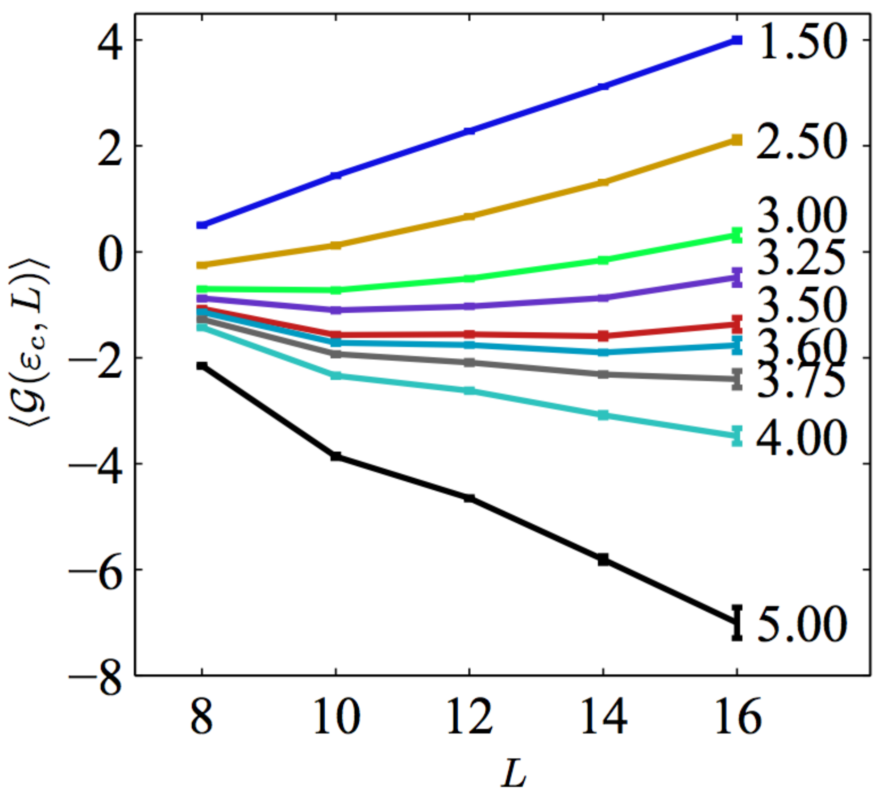

Ergodic phase: $\mathcal{O}_{nm} \sim 1/\sqrt{\mathcal{D}}$

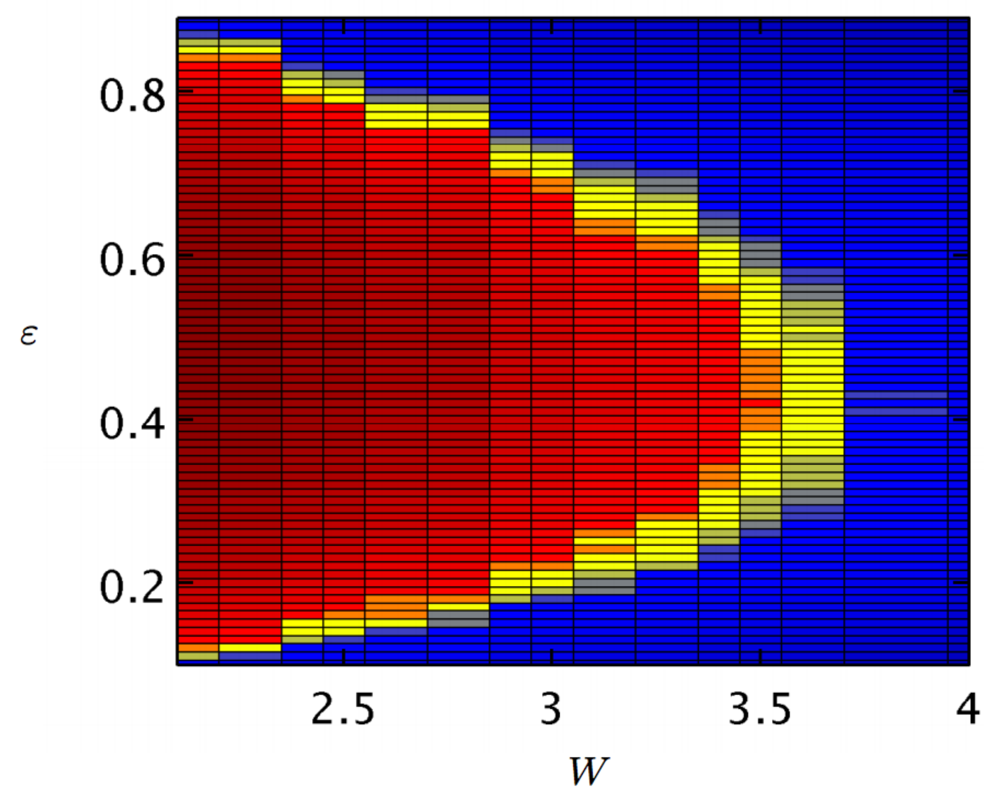
$\delta \sim 1/\mathcal{D}$

$\mathcal{G}(L) \sim L$

MBL phase: $\mathcal{O}_{nm}/\delta \propto \exp(-\kappa L)$

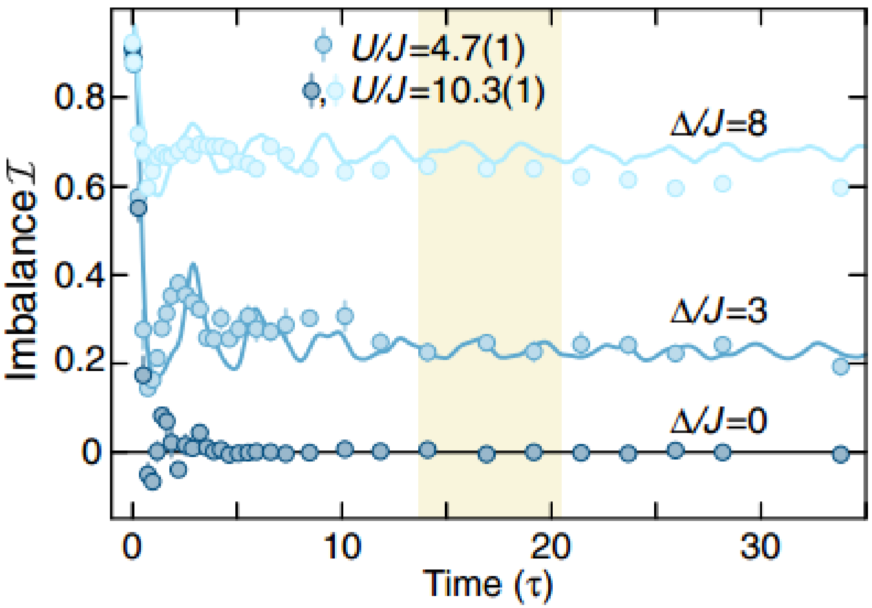
(Similar conclusions for $\langle \ln \frac{\frac{\partial^2 E_n}{\partial \phi^2}}{\delta}\rangle$ )



# More consequences of Local Integrals of Motion

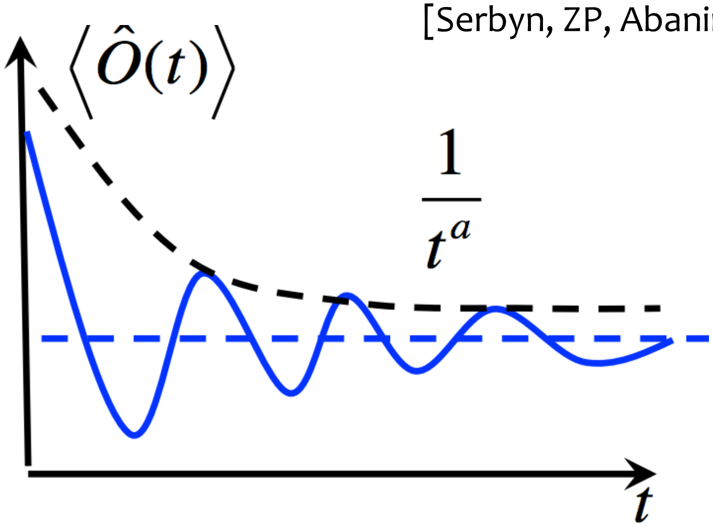
- **Consequence 1: System does not relax**




[M. Schreiber *et al*, Science **349**, 842 (2015)]

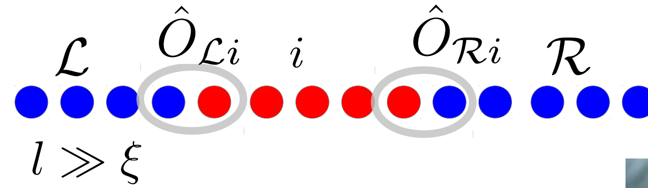
- **Consequence 3: Local observables decay as power law in time**

[Serbyn, ZP, Abanin, '14]



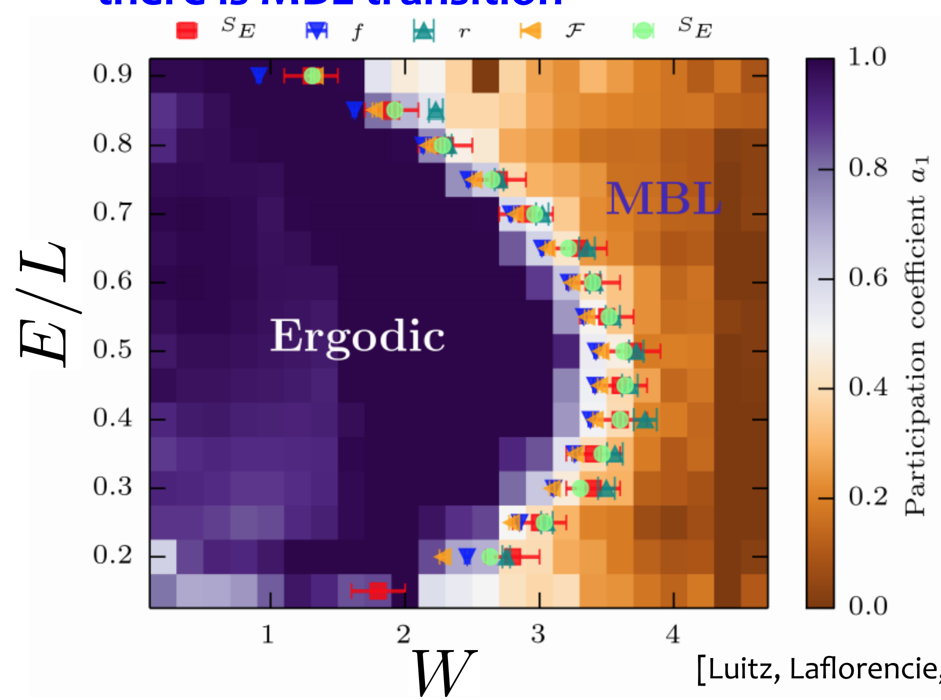

- **Consequence 2: Area law entanglement in all states**

[Bauer, Nayak '13]




Can be efficiently simulated on classical computers using matrix product state techniques

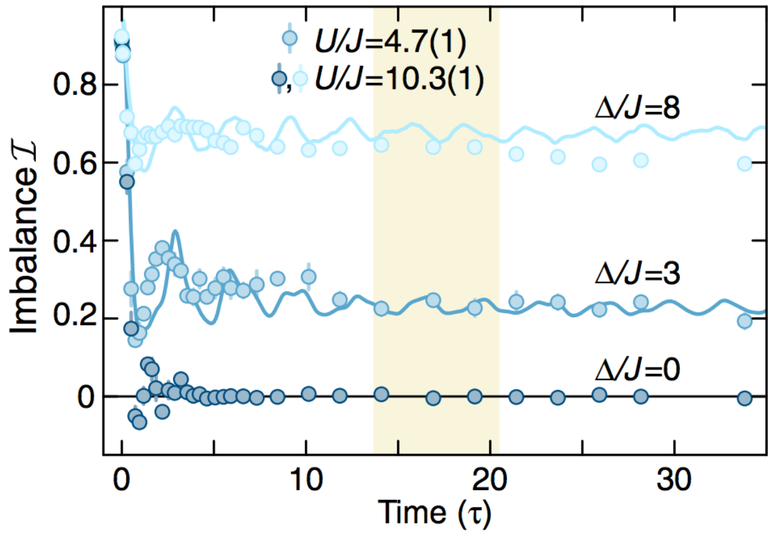
- **Consequence 3: As disorder is reduced, LIOMs delocalize, transport is restored, and there is MBL transition**




[Luitz, Laflorencie, Alet '14]

# Experiments

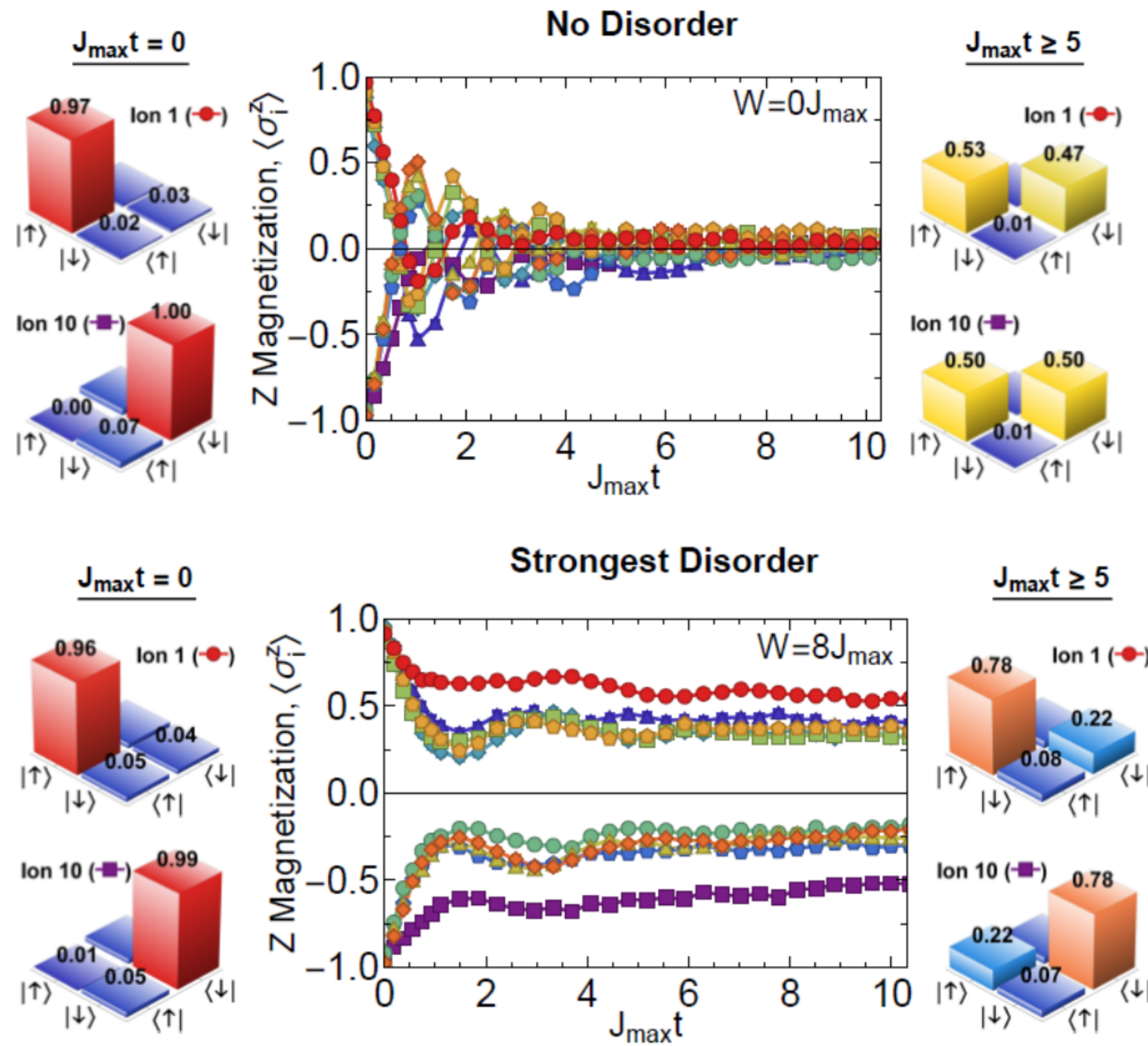
$$\mathcal{I} = \frac{N_e - N_o}{N_e + N_o}$$




Fermi Hubbard + $\sum_{j,\sigma} \Delta \cos(2\pi\beta j)\hat{n}_{j,\sigma}$
$J, U$

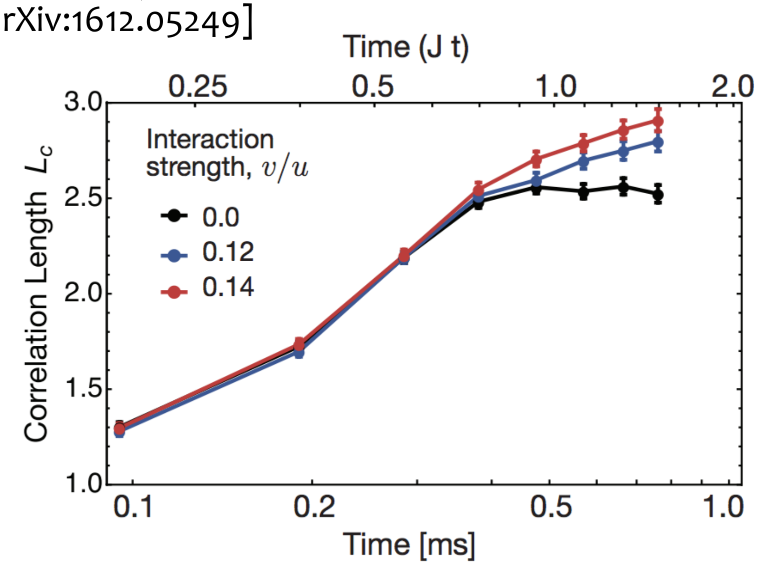
[Schreiber *et al.*, Science **349**, 842 (2015)]

$$H = \sum_{i,j} \frac{\sigma_i^x \sigma_j^x}{|i-j|^\alpha} + \frac{B}{2}\sum_i \sigma_i^z + \sum_i D_i \sigma_i^z$$




[Smith *et al*, Nature Physics 12, **907** (2016)]

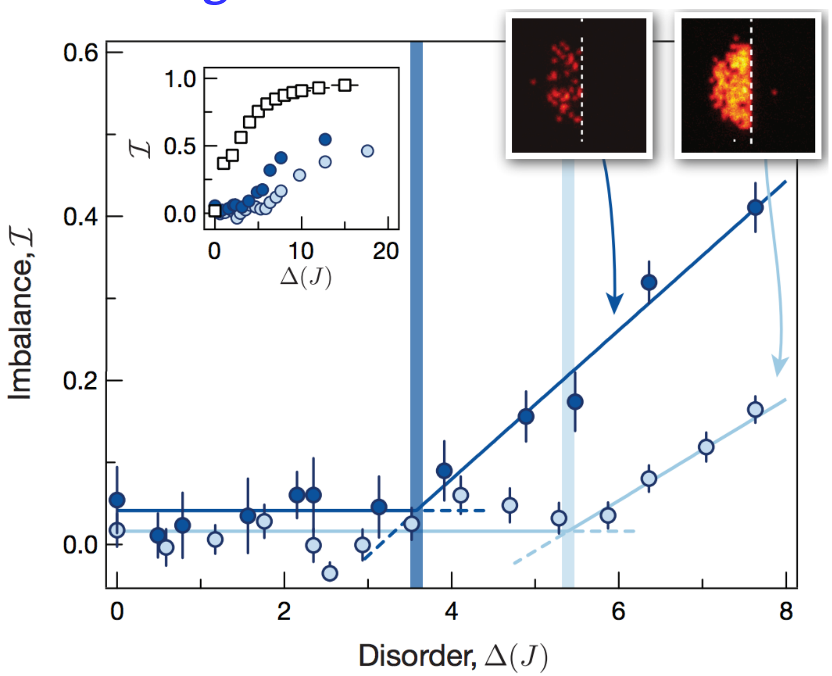
Nuclear spins $[Ca_5(PO_4)_3F]$
[Wei *et al.*,
arXiv:1612.05249]




Hopping + $\sum_j h_j \sigma_j^z + \sum_j v(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - 2\sigma_j^z \sigma_{j+1}^z)$

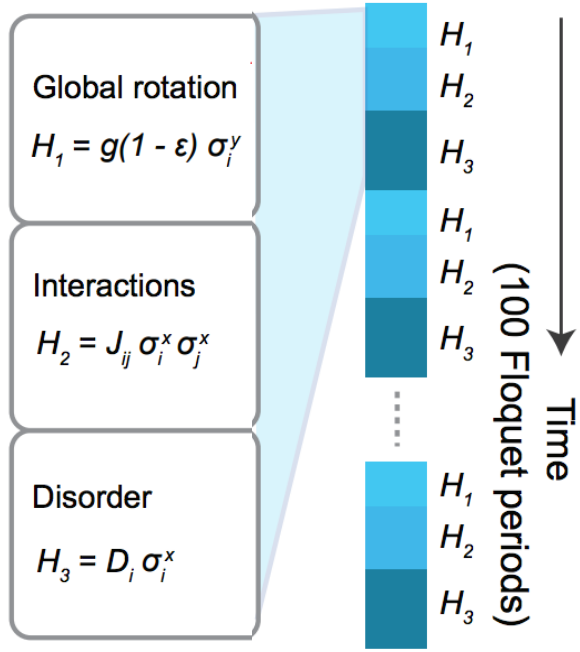
# New directions

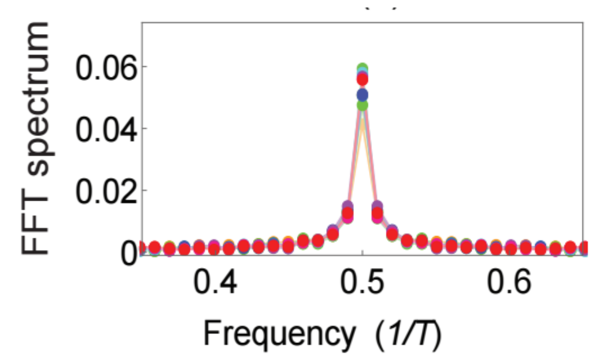
## MBL in **higher dimensions**




2D: [Choi *et al.*, Science **352**, 1547 (2016)]

## MBL in **Floquet systems, time crystals**

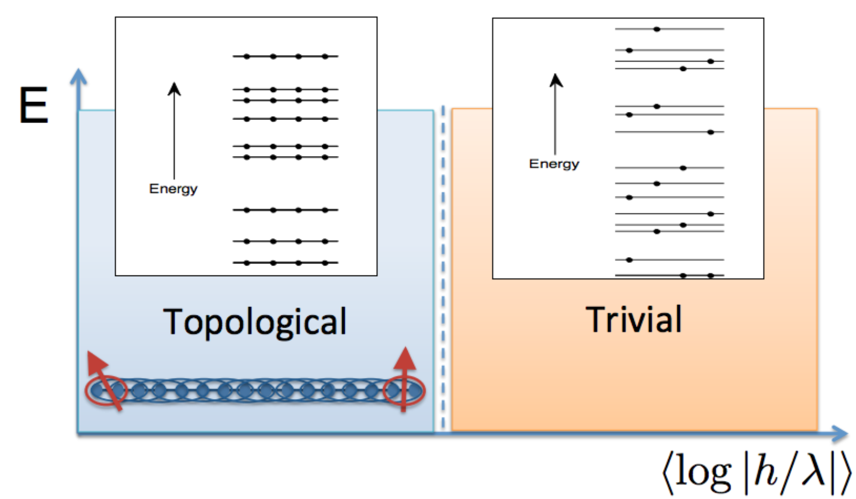



$\varepsilon = 0.03,\ 2\pi J_0\, t_2/(W\, t_3) = 0.07$

## MBL protected **(topological) order**

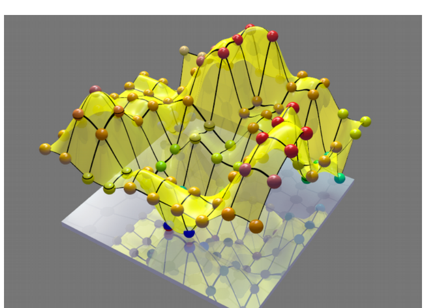
[Huse, Nandkishore, Oganesyan, Pal, Sondhi, PRB **88**, 014206 (2013)]




[Bahri *et al.*, Nat. Commun. 7341 (2015)]

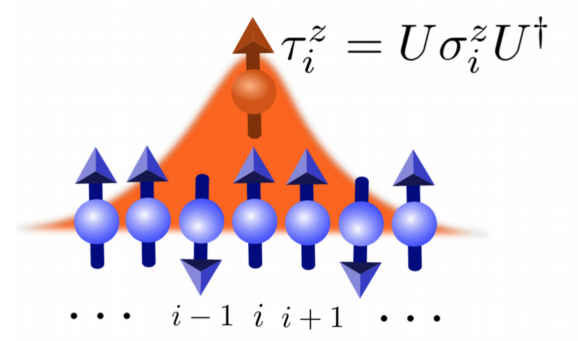
## MBL in systems **without disorder**
## Quantum glasses?

[M. Schiulaz, M. Muller, '13; T. Grover and M. Fisher, '13; de Roeck/Huveneers, '13; Hickey, Genway, Garrahan, '14; Yao *et al.*, '14; ZP, Stoudenmire, Abanin '15, ...]



# Summary

- Anderson localization stable w.r.t. short-range interactions/finite density of particles in 1D
- "Many-body localization" is a consequence of local integrability




- Many-body localized phases have many unique properties which make them different from both Anderson insulators and ergodic systems

| Thermal phase | Anderson | MBL |
|---|---|---|
| Memory of initial condition hidden in global operators | Some memory of initial condition persists | Some memory of initial condition persists |
| ETH true | ETH false | ETH false |
| Generally non-zero DC conductivity | Zero DC conductivity | Zero DC conductivity |
| Volume law entanglement in eigenstates | Area law entanglement in eigenstates | Area law entanglement in eigenstates |
| Power law spreading of entanglement | Finite spreading of entanglement | Logarithmic spreading of entanglement |
| Local magnetization decays exponentially | Local magnetization does not decay | Local magnetization decays as power law |

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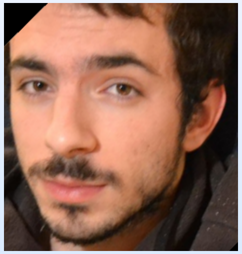

Alexios Michailidis
Leeds

Chris Turner
Leeds

Other collaborators:

Thomas O'Brien
Anushya Chandran
Eugene Demler
Sarang Gopalakrishnan
Francois Huveneers
Isaac Kim
Michael Knap

Chris Laumann
Misha Lukin
Pedro Ponte
Tomaz Prosen
Miles Stoudenmire
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Norman Yao,
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